BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME XLIII, NUMBER 8

AUGUST, 1937

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1937

Whole No. 455 $9.00 a Year

PUBLISHED MONTHLY

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4., section 528, P. L. and R., authorized May 9, 1935.

PERIODICAL PUBLICATIONS
OF THE
AMERICAN MATHEMATICAL SOCIETY

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY

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MODERN METHODS OF ANALYSIS IN POTENTIAL THEORY*

BY G. C. EVANS

I shall take the liberty of interpreting the title in terms of a short discussion of certain problems that have been of recent interest, particularly to younger mathematicians, namely, the discontinuous boundary value problems of the Dirichlet and Neumann types, the mixed problem, and finally the generalizations of the potential integral itself to others in which the integrand $1/r$ is replaced by $1/r^\alpha$, $0<\alpha<3$. In this way my review will serve as a continuation of that of the late Professor Kellogg, but will diverge from it in the sort of problem to be considered.†

I. Dirichlet and Neumann Problems

1. *Plane Regions.* It is a natural generalization of Poisson's integral to write it in the form of a Stieltjes integral,

$$(1)\qquad u(r,\theta) = u(M) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\phi-\theta)}\, dF(\phi),$$

where $F(\phi)$ is periodic and of bounded variation. If the $dF(\phi)$ is replaced by $df(e)$ or $f(de)$ the integral is interpreted as integration with respect to an additive function of sets on the circumference, or in other words, with respect to an arbitrary distribution of positive and negative mass on the circumference, finite in total absolute amount.

The integrand $(a^2-r^2)/[2\pi(a^2+r^2-2ar\cos(\phi-\theta))]$ may also be written in terms of the Green's functions $g(M, P)$ or its conjugate $h(M, P)$, in which P is the point on which is performed the integration. In fact, (1) may be written in the form

$$(2)\qquad u(M) = \int_C \frac{dh(M,P)}{dh(O,P)}\, df(e_P)$$

* An address delivered by invitation of the Program Committee, at the Cambridge Meeting of the Society, September 5, 1936.

† O. D. Kellogg, *Recent progress with the Dirichlet problem*, this Bulletin, vol. 32 (1926), pp. 601–625.

in which a factor a has been absorbed in the $f(e)$, and, for a smooth curve,

$$\frac{dg(M, P)}{dn_P} \Big/ \frac{dg(O, P)}{dn_P} = \frac{dh(M, P)}{ds_P} \Big/ \frac{dh(O, P)}{ds_P} = \frac{dh(M, P)}{dh(O, P)}.$$

In this form the expression (2) is evidently invariant under conformal transformation, and, if the boundary is regarded as constituted of "prime ends" as elements, ordered according to the values of $h(O, P)$, may be applied to any simply connected region and its boundary (provided that the latter does not consist of a single point). But it is easily seen that a necessary and sufficient condition for (1) is that $u(M)$ be the difference of two functions harmonic and not negative in the circle; hence the same class of functions with reference to the general domain is represented by (2).*

But also, except for an additive constant in the integrand, the form (1) may be rewritten in terms of the familiar expression for the potential of a double layer on the circumference. Such an expression, involving a direction of doublets, demands however a certain smoothness in the curve itself, and therefore cannot be extended to general boundaries. Nevertheless, for regions bounded by curves which are sufficiently smooth, it represents again the same class of functions.

Curves of particular importance with respect to such problems are those "of bounded turning," introduced by Radon.† These are curves which have a forward and backward tangent at each point, such that the total variation of the angle turned through by a tangent remains bounded as the whole curve is traced. These curves provide a natural basis for the treatment, in terms of an extension of the theory of integral equations, of mass distributions on them and the related Dirichlet and Neumann problems, and thus generalize to a natural limit the earlier treatment of Plemelj in terms of general mass distribu-

* G. C. Evans, *The Logarithmic Potential*, American Mathematical Society Colloquium Publications, vol. 6, New York, 1927.

† J. Radon, *Über die Randwertaufgaben beim logarithmischen Potential*, Sitzungsberichte der Akademie der Wissenschaften in Wien, vol. 128 (1919), pp. 1123–1167.

tions.* The class of harmonic functions which is typical is rather the class of bounded than merely positive ones. But the analysis involves a much more searching study of linear functional operations than the treatment of continuous density distributions in terms of the Fredholm theory of integral equations, and thus seems, with the work of Fréchet and F. Riesz, to be the basis of much of the modern theory of linear operations. As for the corresponding problem in space, no one seems to have found the proper generalization of curves of bounded turning, the obvious one in terms of the Gaussian sphere not being adequate.

2. *Regions in Space with Smooth Boundaries.* In three dimensions regions convenient for this sort of study are those with a bounding surface which has a tangent plane at every point and is such that the angle between its normals at nearby points P, Q is dominated by an expression $N \cdot \overline{PQ}$, where N is independent of P and Q. The potential of a double layer has the form

$$U(M) = \int_s \frac{\cos(r, n_P)}{r^2}\, d\nu(e_P), \qquad r = \overline{MP}, \tag{3}$$

and represents again the class of functions which are differences of two positive harmonic functions. These are given also by a formula analogous to (2). In fact, the connection between $\nu(e)$ and $f(e)$ is stated in terms of Stieltjes integral equations. Moreover the class of functions furnishes unique solutions of the boundary value problems in which is given on s an additive function which is the limit of the integral of $U(M)$ over arbitrary pieces of smooth surfaces approximating to s. The function $U(M)$ itself approaches a limiting value as M tends to a point Q of s, for almost all Q, if M remains within a cone with vertex Q and axis the normal at Q, the cone lying within the region. Analogous considerations apply to the Neumann problem and potentials of a single layer. The data given in the boundary problem are the limiting values of the flux, or integral of the normal derivatives, over arbitrary pieces of the approximating surfaces.†

* Plemelj, *Potentialtheoretische Untersuchen*, Leipzig, 1911.

† G. A. Garrett, *Necessary and sufficient conditions for potentials of single and double layers*, American Journal of Mathematics, vol. 58 (1936), pp. 95–129. Reference is given in this memoir to work of H. E. Bray, E. R. C. Miles, C. de la Vallée Poussin, and myself. Some of this same circle of ideas has been

3. *Boundaries Consisting of Regular Points.* The form (2), as we have said, we may expect to be carried much further in the direction of general boundaries than the form (3). In two dimensions there was no restriction on the (simply or finitely connected) domain. In three dimensions the question remains as to how far may be carried the representation of functions which are the differences of not negative functions harmonic in the domain—or more simply of functions positive and harmonic in the domain.

A recent statement of this problem is given by Maria and Martin,* direct in the sense that it does not depend on conformal mapping in its analysis. However, it involves the mass function obtained by the sweeping out of unit mass, which is equivalent to the limit on the boundary of the indefinite integral of the normal derivative of the Green's function, and thus in two dimensions to the function $h(M, P)$. What seems essentially novel in three dimensions is the use of the derivative of one positive function of point sets with respect to another,† which yields the analog of the expression $dh(M, P)/dh(O, P)$.

Let then $m(e, M)$ denote the mass distribution obtained by sweeping out unit mass located at a point M of the domain S onto the boundary s of the domain. This is the function whose potential is the solution of the continuous Dirichlet problem corresponding to boundary values $1/r$ (where $r = MQ$, Q a boundary point); the distribution function is for given e itself harmonic in M, and provides the formula

$$u(M) = \int_s u(P)dm(e_P, M)$$

for the solution of the Dirichlet problem.‡ What turns out to

developed by G. Gunther, *Sur une application des intégrales de Stieltjes au problème de Neumann*, Comptes Rendus de l'Académie des Sciences, vol. 189 (1929), pp. 447–450; *Sur les Intégrales de Stieltjes et leurs Applications*, Leningrad, 1932.

* A. J. Maria and R. S. Martin, *Representation of positive harmonic functions*, Duke Mathematical Journal, vol. 2 (1936), pp. 517–529.

† P. J. Daniell, *Stieltjes derivatives*, this Bulletin, vol. 26 (1920), pp. 444–448.

‡ C. de la Vallée Poussin, *Extension de la méthode du balayage de Poincaré, et problème de Dirichlet*, Annales de l'Institut Henri Poincaré, vol. 2 (1932), pp. 169–232.

be the representation formula for the class of positive harmonic functions is the following:

$$u(M) = \int_s f(P, M)d\mu(e_P), \tag{4}$$

where $\mu(e)$ is a distribution of positive mass on s, and where the function

$$f(P, M) = \frac{dm(e, M)}{dm(e, M_0)}$$

is the above mentioned set function derivative taken at the point P on s. With the $f(P, M)$ thus fixed, with M_0 also chosen, to each $u(P)$ there corresponds a $\mu(e)$ and vice versa.

This representation is contingent on the character of the domain and its boundary. The domain need not be simply connected. Every boundary point is assumed to be regular with respect to the Dirichlet problem. It is also assumed that the "principle of Picard" is valid for every point of s, that is, a function positive and harmonic in S which vanishes continuously at every point of the boundary s except at one point Q is determined except for a multiplicative constant.

In order to illustrate a situation where this principle is not valid, consider a plane domain T consisting of a circle cut along one radius, and let Q be an interior point of this radius. If M is a point of T there will be two values of $dg(M, Q)/dn$, say $g_1(M, Q)$ and $g_2(M, Q)$, according as the derivative refers to one side of the cut or the other. The two harmonic functions $g_1(M, Q)$ and $g_2(M, Q)$ are not negative and vanish as M tends towards the boundary, except at Q, and each vanishes for different modes of approach to Q. Hence they cannot be linearly dependent.*

Now let $E(u, S)$, with $u \geqq 0$ in S and harmonic, be the set of points Q of s where lim sup $u(M) > 0$ as M tends to Q. Let u_n

* In two dimensions the order of multiple boundary points is treated by C. Carathéodory with the invention of prime ends. The corresponding Dirichlet problem is discussed in G. C. Evans, *The Logarithmic Potential*, loc. cit., p. 82.

A significant generalization to three dimensions has been developed by F. W. Perkins, *The Dirichlet problem for domains with multiple boundary points*, Transactions of this Society, vol. 58 (1925).

be a sequence of functions of the above kind with limit function u. The third requirement on the boundary is that if $E(u_n, S)$ is contained in a closed set B of s for all n, the same property holds for u.

Finally, a fourth requirement is imposed. For the given domain S there must exist a sequence of nested domains S_n, satisfying the first three requirements, with the following property. If M_0 is a point of S, and B a closed set in $S+s$, and if $u(M)$ is less than some fixed bound at M_0, has as domain some S_n, and is such that $E(u, S_n)$ is contained in B, then u is bounded in a neighborhood of any point not in B, uniformly with respect to n.

These four conditions guarantee the possibility and uniqueness of the representation (4). On the other hand, if the representation is unique, and in all cases $u(M)$ takes on the value zero at any point of s which does not lie on the nucleus of μ (that is, at a point which can be surrounded by a sphere which contains no mass), the first three conditions follow. In particular, in the plane, it can be shown (by means of conformal transformations) that these four conditions are satisfied, for instance, by the domains bounded by simple closed curves.

4. *Poisson's Equation.* A positive harmonic function is not always expressible as a Newtonian potential of a distribution of positive mass, since even in the unit sphere a potential of a double layer is required. But a function which is harmonic in the whole of space except on a set where it becomes positively infinite is necessarily the Newtonian potential of a positive mass distributed on a set of capacity zero. Similarly a necessary and sufficient condition that a function, harmonic in a domain, be the potential of a distribution of positive mass on the boundary is evidently given immediately in terms of the Poincaré sweeping-out process.

Related to such notions is the question of boundary value problems for Poisson's equation. But for the same degree of generality with respect to mass distributions, the Laplacian

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u$$

must be considered as some sort of differential operator as a

whole. Zaremba's definition,* where $\nabla^2(u)$ stands for

$$u(x+h, y, z) + u(x-h, y, z) + u(x, y+h, z) + \cdots + u(x, y, z-h) - 6u(x, y, z),$$

allows one to treat Poisson's equation $\nabla^2 u = -4\pi f(x, y, z)$ without imposing a Hölder condition on the density $f(x, y, z)$ of the mass distribution. Still more general is the expression of Poisson's equation in the form

$$\int_s \frac{du}{dn}\, ds = 4\pi m(s),$$

where $m(s)$ is the regularized function of surfaces corresponding to an arbitrary distribution of mass $m(e)$.†

II. The Mixed Problem

5. *A Uniqueness Theorem.* The difficulty in the mixed problem, in which limiting values which the harmonic function itself takes on are given on a part of the boundary, and the values which the limit of the normal derivative (or indefinite integral of the normal derivative takes on) are given on the rest, lies in the specification of a convenient class of functions in which the solutions are unique. This difficulty seems to underlie also the application of the method of integral equations to the problem, for these carry with them some implication of function class. Accordingly we content ourselves with the working out of a fairly simple example of the problem, merely citing some of the bibliography of other studies.‡ The solution will be given in terms of the simpler Dirichlet problem.

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* S. Zaremba, *Contribution à la théorie d'une équation fonctionelle de la physique*, Rendiconti del Circolo Matematico di Palermo, vol. 19 (1905), pp. 140–150.

H. Petrini, *Les dérivées premières et secondes du potentiel*, Acta Mathematica, vol. 31 (1908), pp. 127–332.

† G. C. Evans, *Discontinuous boundary value problems of the first kind for Poisson's equation*, American Journal of Mathematics, vol. 51 (1929), pp. 1–18.

‡ S. Zaremba, *Sur un problème mixte relatif à l'équation de Laplace*, Bulletin de l'Académie des Sciences de Cracow, 1910, pp. 313–344.

L. Lichtenstein, *Randwertaufgaben der linearen elliptischen Gleichungen*, Journal für die reine und angewandte Mathematik, vol. 143 (1913); see pp. 93–105.

H. Villat, *Sur la résolution de certaines équations intégrales, et sur quelques problèmes qui s'y rattachent*, Acta Mathematica, vol. 40 (1916); see pp. 162–167.

Let S be an open circular region, s the circumference, O the center, and F a closed set of points on s of positive capacity.* Denote by $s-F$ the sum of arc-intervals $I_1, I_2, \cdots$, which is complementary to F with respect to s, and by CF the complement of F with respect to the whole plane. Let $f(P)$ be a function which is defined and continuous on F, and $h(e)$ a completely additive function of sets on $s-F$, that is, completely additive on each interval I_k and such that its total variation is bounded on $\sum I_k$.

We note first the following uniqueness theorem:

THEOREM. *There cannot be more than one function $u(M)$ which is harmonic in S and*

(a) *is bounded in a neighborhood of F;*

(a′) *takes on the values $f(P)$ as M in S tends to P in F, except for a subset of points P of zero capacity;*

(b) *is such that $\int_0^A (du/dn)\,ds$ is bounded as A in S tends to P in any closed subinterval of each I_k;*

(b′) *is such that $\int_A^B (du/dn)\,ds$ tends to $h(PQ)$ where PQ is an arc interval and A tends to P, B to Q in an interval I_k of $s-F$, except for a denumerable infinity of points P, Q.*

In fact, if $w(M)$ is the difference of two possible $u(M)$, and $v(M)=\int_0^M (dw/dn)\,ds$ its conjugate function, the function $v(M)$ is bounded as M tends to the boundary in any closed interval contained in an I_k and takes on a constant value c_k except at a denumerable infinity of points P of I_k. It is therefore harmonic except for removable discontinuities at every point of I_k and uniquely extensible across it, by means of the formula

$$v(M) - c_k = c_k - v(M'), \qquad M' \text{ inverse to } M \text{ in } s.$$

It follows that $w(M)$ can be made harmonic at every point of I_k and that $w(M)$ is extensible across the cut I_k by means of a formula

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G. C. Evans and R. N. Haskell, *The mixed problem for Laplace's equation in the plane. Discontinuous boundary values*, Proceedings of the National Academy, vol. 16 (1930), pp. 620–526.

H. Hornich, *Lösung einer vermischten Randwertaufgabe der Potentialtheorie durch hyperelliptische Integrale*, Monatshefte für Mathematik und Physik, vol. 39 (1932), pp. 107–128.

* A bounded set is of zero capacity if it will bear no positive mass distribution with a bounded potential. It is of positive capacity if it will bear a positive mass distribution with a bounded potential; see de la Vallée Poussin, loc. cit.

$$w(M') - w(M) = \gamma_k, \qquad \gamma_k \text{ a constant.}$$

But since $w(M)$ is harmonic on I_k, we have

$$w(M') = w(M),$$

and the function so defined is single valued and harmonic in the whole plane except on F, and bounded. It follows then that if $w(M)$ is not identically 0, the set of points P on F where $\limsup_{M=P} w(M) > 0$ or $\liminf_{M=P} w(M) < 0$ contains a closed set of positive capacity, as M in CF tends to P.*

This statement, however, contradicts our hypothesis. In fact, let Q be a point of F not in the exceptional set for $w(M)$ as defined in the hypothesis (a') of the theorem. Let $M_1, M_2, \cdots, M_n, \cdots$ be a sequence of points in CF tending to Q. To these there corresponds a sequence N_n in S, tending to Q, where

$N_n = M_n$ if M_n is in S,
$N_n = M_n'$ if M_n is in $C(S+s)$,
N_n lies in S sufficiently close to M_n, if M_n lies on an interval I_k.

The N_n may therefore be chosen so that $0 = \lim_{n=\infty} u(N_n) = \lim_{n=\infty} u(M_n)$. Hence, for M in CF, $\lim_{M=Q} w(M) = 0$ except for Q in a subset of F of zero capacity.

We conclude then that $w(M) \equiv 0$ and $u(M)$ is unique.

6. *Construction of a Solution.* Let $u(M)$ be the solution of the Dirichlet problem for the infinite region of boundary F corresponding to the given values $f(P)$. Such a solution exists, and takes on the boundary values $f(P)$ continuously, except at the irregular points of F. These in fact constitute a subset (not necessarily closed) of zero capacity.

The boundary values are taken on as M tends to P from CF, and therefore, in particular, from S. Moreover we note that $u_1(M) = u_1(M')$, since the problem is unchanged by inversion with respect to the circumference s.

In particular, let $g_F(A, B)$ be the Green's function for the domain CF. Then again, with consideration of the inversion,

$$g_F(A, B) = g_F(A', B'). \tag{5}$$

We shall consider briefly the properties of the function

$$u_2(M) = -\frac{1}{\pi}\int_{s=F} g_F(M, P)\,dh(e_P) = u_2(M'). \tag{6}$$

* O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929; see p. 335.

Let A, B be two points of S, which we shall allow ultimately to tend to two points X, Y respectively of the same interval I_k of $s-F$, which are points of continuity of $h(e)$, the polar angle of X being less than that of Y; and let l be a curve joining A, B, with a continuously turning tangent. We shall choose as a direction of the normal n at a point of l that direction which points toward the center when l has been deformed into XY, and define $ds = |ds|$. When A and B have become near to X, Y the flux

$$(6') \qquad N = \int_A^B \frac{du}{dn}\,ds = -\frac{1}{\pi}\int_{s-F} dh(e_P)\int_A^B \frac{dg_P}{dn_M}\,ds$$

may be evaluated in simple terms.

In fact, if we make an inversion in the circle s, letting n' be the direction corresponding to n by the inversion, we have

$$2N = \int_A^B \frac{du}{dn}\,ds + \int_{A'}^{B'} \frac{du}{dn'}\,ds' = \int_A^B \frac{du}{dn}\,ds + \int_{B'}^{A'} \frac{du}{dn'}\,ds',$$

$$-2N = \frac{1}{\pi}\int_{s-F} dh(e_P)\left\{\int_A^B \left[\frac{d}{dn}\log\frac{1}{MP} + \frac{d}{dn}\gamma(M,P)\right]ds + \int_{B'}^{A'}\left[\frac{d}{dn'}\log\frac{1}{M'P} + \frac{d}{dn'}\gamma(M',P)\right]ds'\right\},$$

where $\gamma(M, P)$ is harmonic and bounded, with its derivatives, as long as M is at a distance from F which is at least as great as some constant $\delta>0$. We can therefore take A and B near enough to X and Y so that the integrals of $|d\gamma(M,P)/dn_M|$ along radii from A to A' and B to B' will be as small as we please. Hence

$$-2N = \frac{1}{\pi}\int_{s-F} dh(e_P)\left\{\int_A^B \frac{d}{dn}\log\frac{1}{MP}\,ds + \int_{B'}^{A'}\frac{d}{dn'}\log\frac{1}{M'P}\,ds'\right\} + \eta(A,B),$$

where given $\epsilon>0$, we can take A, B close enough to X, Y so that

$$|\eta(A,B)| < \epsilon.$$

In fact the integral of the exterior normal derivative of $\gamma(M, P)$ around the closed contour $ABB'A'A$ is zero. Moreover

$$\int_A^B \frac{d}{dn} \log \frac{1}{MP}\, ds = -\measuredangle BPA,$$

$$\int_{B'}^{A'} \frac{d}{dn'} \log \frac{1}{M'P}\, ds' = -\measuredangle A'PB'$$

by direct computation.

Accordingly

$$\lim_{\substack{A=X \\ B=Y}} N = \frac{1}{2\pi} \lim_{\substack{A=X \\ B=Y}} \int_{s-F} (\measuredangle BPA + \measuredangle A'PB')dh(e_P)$$

and since the integrand remains bounded as A tends to X and B to Y, and X, Y are not point masses of $h(e)$, it follows that

$$\lim_{\substack{A=X \\ B=Y}} N = h(I_{XY}), \tag{7}$$

where I_{XY} stands for the arc interval X, Y. There are, of course, at most a denumerable infinity of point masses of $h(e)$.

The function

$$u(M) = u_1(M) + u_2(M)$$

is a type of solution of the mixed boundary value problem. In order to bring it into the class of solutions to which the uniqueness theorem applies, we must, however, make restrictions on the set function $h(e)$ or the set F, or both. The function $u_1(M)$ gives no trouble. The function $u_2(M)$, if $h(e)$ is not restricted, may however become infinite at points which are ends of intervals I_k, and even at certain other points—if for example F is a perfect non-dense set on s. The points of F which are limit points of $s-F$ need not constitute a set of zero capacity.

On the other hand, if $h(e)$ is absolutely continuous and its derivative is bounded, the function $u_2(M)$ vanishes continuously at every regular point of F; for $g_F(M, P)$ is the Green's function for F, and it vanishes at regular points of F, and, since $0 \leqq g_F(M, P) \leqq \log 1/\overline{MP}$, the absolute continuity of $\int_{s-F} g_F(M, P)h'(P)ds_P$ is uniform as M approaches a point Q which is in F.

THEOREM. *If $h(e)$ is absolutely continuous, with a bounded derivative, the function $u(M) = u_1(M) + u_2(M)$, where $u_1(M)$ and $u_2(M)$ are the functions defined in this section, is a solution of the mixed boundary value problem and belongs to the class of functions to which the uniqueness theorem of* §5 *applies.*

Another special case arises if F consists merely of a finite number of closed intervals, and $h(e)$ is absolutely continuous with a bounded derivative in a neighborhood of the end points of each interval of $s-F$, so that $u_2(M)$ remains bounded in a neighborhood of F.

Hornich considers this latter case, with continuously given boundary data, and also generalizes the problem by considering the case in which along the real axis values of the quantity

$$\frac{\partial u}{\partial y}+\lambda_i\frac{\partial u}{\partial x}=h$$

are assigned on certain intervals (b intervals) of the boundary, while the values of u itself are assigned on other intervals (a intervals).* His method is likewise an extension of his method used in the earlier problem, namely, the construction of a Green's function as the real part of a function of a complex variable, with branch points at the ends of the intervals. In this case, however, the function is not uniquely determined, if it exists, unless the a- and b-intervals alternate.

A related problem is the three dimensional one, discussed by Bouligand, Giraud, and Delens,† where the boundary values of a slanting derivative are given. If the direction of the given derivative is nowhere tangent to the boundary, a uniqueness theorem can be deduced inmediately. For the difference of two such solutions would be one with a zero slanting derivative, and, for sufficiently smooth boundaries, if this solution were not identically zero, the derivative could not be zero at the points of the boundary at which the function took on its highest and lowest bounds. By construction of a generalized Green's function (of which the Laplacian is not zero), based on a "potential of an oblique double layer," the solution may be written explicitly, provided that the given derivative is subject to a certain integral condition of compatibility. The differential geometry of the situation in the case where the homogeneous problem

* H. Hornich, *Die allgemeine vermischte Randwertaufgabe der ebenen Potentialtheorie*, Monatshefte für Mathematik und Physik, vol. 39 (1932), pp. 455–460, and vol. 41 (1934), pp. 7–19.

† G. Bouligand, G. Giraud et P. Delens, *Le Problème de la Dérivée Oblique en Théorie du Potentiel*, Paris, 1935.

has solutions, the given direction being somewhere tangent to the boundary, may be developed in terms of a special theory of congruences.

III. Potentials in Space, not Newtonian

7. *Form of Integral.* The potentials to be considered are of the form

$$(8) \qquad u(M) = \int_F \frac{1}{r^\alpha}\, dm(e_P), \qquad r = \overline{MP}, \qquad 0 < \alpha < 3,$$

in which the mass is located on a closed bounded set F in space. These do not seem to be subject to differential equations—excluding fractional derivatives—unless the mass is localized on a set of special character. For instance, if the mass lies in the plane $x=0$, the potential for values of $x \neq 0$ satisfies the equation

$$\nabla^2 u + \frac{4\alpha - 1}{x} \frac{\partial u}{\partial x} = 0,$$

which is called Green's equation by Pierre Humbert.* The same remark applies to potentials of more general form where $1/r^\alpha$ is replaced by a suitable function $\Phi(r)$.

Of special interest, however, are the results which are independent of particular forms of mass distribution, and thus do not depend on the methods of differential equations. In this direction, the recent thesis of Otto Frostman† deals with the problem of capacity and conductor potential, as related to the transfinite diameter of Fekete and Pólya and Szegö,‡ and makes application to the complex variable on the two dimensional sphere. The present section may be regarded as an introduction to the work of Frostman. The problem which we take up will be that of the conductor potential. Our methods will differ only slightly from those of Frostman and we shall only slightly extend his results.

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* Humbert, *Potentiels et Prépotentiels*, Paris, 1936.

† O. Frostman, *Potentiel d'Équilibre et Capacité des Ensembles*, Lund, 1935.

‡ Pólya and Szegö, *Transfiniter Durchmesser ebener und räumlicher Punktmengen*, Journal für die reine und angewandte Mathematik, vol. 165 (1931), pp. 4–49.

8. *Minimum Energy.* The intrinsic energy, or simply the energy of a distribution, is defined as the iterated integral

$$I = \int_F \int_F \frac{1}{MP^\alpha} d\sigma(e_P) d\sigma(e_M). \tag{9}$$

It is a quantity which Frostman shows to be positive, if finite, when $\sigma(e)$ is any additive function of point sets, not necessarily positive; in fact $I=0$ only if $\sigma(e)\equiv 0$. This result is obtained by rewriting I as the integral over space of a square of a potential, making use of the identity

$$\int_W \frac{1}{MP^\beta} \frac{1}{PQ^\gamma} dP = \text{const.} \frac{1}{MQ^\alpha}, \qquad \begin{cases} \alpha = \beta + \gamma - 3, \\ W = \text{entire space}, \end{cases}$$

and taking $\beta=\gamma$. An equilibrium distribution or conductor distribution, if it exists, is a distribution of a given total mass on F in such a way as to give to I its minimum value; the resulting potential is the equilibrium or conductor potential.

For convenience in this exposition it will be said that a set F is of *zero capacity* if the energy of any distribution of positive mass located on it is $+\infty$, and of *positive capacity* if it is possible to distribute positive mass on it in such a way that I remains finite.* If F is of positive capacity and $\mu(e)$ and $\lambda(e)$ are two distributions of positive mass on F such that the energies $I(\mu)$ and $I(\lambda)$ are finite, it can be seen that $\mu(e)+\lambda(e)$ is another such distribution, and also that the quantity

$$I(\mu, \lambda) = \int_F \frac{1}{MP^\alpha} d\mu(e_M) d\lambda(e_P)$$

is finite. In fact

$$I(\lambda + \mu) = I(\lambda) + 2I(\lambda, \mu) + I(\mu).$$

Suppose then that we have two closed bounded sets of positive capacity, F_1, F_2, with no point in common, and therefore at a positive distance δ, and that a distribution of positive mass $\mu(e)$ of total mass μ is spread on F_1 and a distribution of negative mass $-\nu(e)$, of total value $-\nu$, is spread on F_2. Does there

* It can be shown that this definition is equivalent to that in which the words "its potential remains bounded" are substituted for the words "I remains finite."

exist such an arrangement for which the energy I takes on a minimum value?

We write

$$
(10)\quad \begin{cases} u(M) = \displaystyle\int_{F_1} \frac{1}{\overline{MP}^{\alpha}}\, d\mu(e_P), \qquad v(M) = \int_{F_2} \frac{1}{\overline{MP}^{\alpha}}\, d\nu(e_P), \\ w(M) = u(M) - v(M), \\ I = \displaystyle\int_{F_1+F_2}\int_{F_1+F_2} \frac{1}{\overline{MP}^{\alpha}}\, d[\mu(e_P) - \nu(e_P)]\, d[\mu(e_M) - \nu(e_M)] \\ \quad = \displaystyle\int_{F_1} w(M)d\mu(e_M) - \int_{F_2} w(M)d\nu(e_M) \\ \quad = I(\mu) + I(\nu) - 2I(\mu, \nu) \\ \quad = \displaystyle\int_{F_1} u(M)d\mu(e_M) + \int_{F_2} v(M)d\nu(e_M) - 2\int_{F_1} v(M)d\mu(e_M). \end{cases}
$$

Let $\mu_n(e), \nu_n(e)$, $n = 1, 2, \cdots$, be a sequence of pairs of distributions with the given total masses for which the corresponding values I_n of I tend to the lower bound I_0 of I. Since $\mu_n(F_1) = \mu$, $\nu_n(F_2) = \nu$, there exists a subsequence of $\{n\}$ for which these mass distributions converge in the weak sense to mass distributions $\mu_0(e), \nu_0(e)$ respectively. And without loss of generality we may regard all the values of n as restricted to this subsequence.

Let I' denote the energy of the distribution $\mu_0(e) - \nu_0(e)$ and $u_0(M)$, $v_0(M)$ the respective potentials of $\mu_0(e)$ and $\nu_0(e)$. We wish to show that $I' = I_0$.

The function $1/\overline{MP}^{\alpha}$ is lower semi-continuous if M and P both lie on F_1 (or F_2) and is continuous if M lies on F_1 and P on F_2. Hence, by the weak convergence,*

$$\liminf_{n=\infty} I(\mu_n) \geqq I(\mu_0), \qquad \liminf_{n=\infty} I(\nu_n) \geqq I(\nu_0),$$

$$\lim_{n=\infty} I(\mu_n, \nu_n) = I(\mu_0, \nu_0),$$

and

$$I_0 \geqq I'.$$

* As in Frostman, loc. cit., p. 33.

But $\mu_0(e)$, $\nu_0(e)$ are admissible distributions, so that

$$I' \geqq I_0.$$

It follows that $I' = I_0$, and the distribution $\sigma_0(e) = \mu_0(e) - \nu_0(e)$ is one for which I takes on its lower bound.

9. *The Equilibrium Distribution.* It remains to find the character of such a conductor distribution $\sigma_0(e) = \mu_0(e) - \nu_0(e)$. To this end we let $\delta\mu(e)$ be a distribution on F_1 for which the energy $I(\delta\mu)$ is finite; it follows that $I(\mu_0 + \delta\mu)$ and $I(\mu_0, \delta\mu)$ are also finite. By substitution, it may be verified that the change of I is given by the formula

$$\begin{aligned}\Delta I = &- 2\int_{F_2}\int_{F_1} \frac{1}{\overline{QP}^\alpha}\, d\nu_0(e_P)d\delta\mu(e_Q) \\ &+ 2\int_{F_1}\int_{F_1} \frac{1}{\overline{QP}^\alpha}\, d\mu_0(e_P)d\delta\mu(e_Q) \\ &+ \int_{F_1F_1} \frac{1}{\overline{QP}^\alpha}\, d\delta\mu(e_P)d\delta\mu(e_Q) \\ &= 2\int_{F_2} w_0(Q)d\delta\mu(e_Q) + I(\delta\mu),\end{aligned} \tag{11}$$

where

$$\Delta I \geqq 0 \text{ if } \int_F d\delta\mu(e_Q) = 0 \text{ and } \mu_0(e) + \delta\mu(e) \geqq 0. \tag{11'}$$

The conditions in (11′) can be met if we write (making use of the principle of the Lagrangian multiplier)

$$\delta\mu(e) = \epsilon\left\{\phi(e) - \frac{\phi(F_1)}{\mu_0(F_1)}\,\mu_0(e)\right\}, \qquad \epsilon > 0, \tag{11''}$$

where $\phi(e)$ is a positive mass distribution on F_1, and ϵ is small enough, $\phi(e)$ being chosen so that $I(\phi)$ is finite. Then $I(\delta\mu)$ is also finite, and from (11′) it is deduced as a necessary condition that

$$\int_{F_1} w_0(Q)d\delta\mu(e_Q) \geqq 0;$$

for otherwise ΔI would be <0 for ϵ small enough. We have then the necessary condition

$$\int_{F_1} w_0(Q)d\left\{\phi(e_Q) - \frac{\phi(F_1)}{\mu_0(F_1)}\mu_0(e_Q)\right\} \geqq 0,$$

which, by defining

$$J_1 = \frac{1}{\mu_0(F_1)}\int_{F_1} w_0(Q)d\mu_0(e_Q),$$

may be written in the form

$$\int_{F_1} [w_0(Q) - J_1]d\phi(e_Q) \geqq 0, \tag{12}$$

where $\phi(e)\geqq 0$ and $I(\phi)$ is finite.

It follows that $w_0(Q)\geqq J_1$ everywhere on F_1 except possibly on a subset whose capacity in our sense is zero.

Denote now by F_1' the closed cover of the subset of F_1 on which the distribution $\mu_0(e)$ lies, so that any point in $F_1 - F_1'$ may be surrounded by a sphere which contains no mass, and that property holds for no point of F_1'. Then we shall see that at every point of F_1', $w_0(Q)\leqq J_1$.

In fact, if P is a point where $w_0(P) > J_1+2\eta$, $\eta>0$, we can surround P by a sphere in which everywhere $w_0(Q)\geqq J_1+\eta$; for $w_0(Q)$ is lower semi-continuous. By hypothesis, this sphere Γ contains a portion of the mass $\mu_0(e)$. Take now $\phi(e) = -\mu_0(\Gamma\cdot e)$. The conditions in (11′) will still be met if ϵ is small enough and $\delta\mu_0(e)$ is given by (11″). But now, for small enough ϵ we shall have $\Delta I<0$, since the left hand member of (12) will be negative. This result is in contradiction with (11′).*

A similar result with inequality signs reversed is obtained for F_2, when we write

$$J_2 = \frac{1}{\nu_0(F_2)}\int_{F_2} w_0(Q)d\nu_0(e_Q).$$

The result to this point may be summarized as follows:

* At this point, our solution answers the problem of the conductor distribution of positive mass on F_1 of given total amount, under the influence of an arbitrary fixed distribution of mass on F_2.

Given a distribution of positive mass on F_1 of total amount μ and of negative mass on F_2 of total amount $-\nu$, F_1 and F_2 being closed bounded sets of positive capacity, with no point in common, there will be distributions $\mu_0(e)$, $-\nu_0(e)$ which will make the total energy (9) a minimum.

There will be two constants, J_1, J_2 such that the potential $w(M)$ of $\mu_0(e)-\nu_0(e)$ will, except for possible subsets of capacity zero, satisfy the relations

$$(13)\qquad \begin{aligned} &w_0(M) \geqq J_1, \quad M \text{ on } F_1; \qquad w_0(M) \leqq J_2, \quad M \text{ on } F_2, \\ &w_0(M) = J_1, \quad M \text{ on } F_1'; \qquad w_0(M) = J_2, \quad M \text{ on } F_2', \end{aligned}$$

where F_1' and F_2' are the portions of F_1 and F_2 respectively which bear the masses $\mu_0(e)$, $\nu_0(e)$.

The integral I thus has the value

$$I_0 = J_1\mu - J_2\nu.$$

We have also the following result:

The minimizing distribution is unique.

Suppose in fact that $m_0(e)$, $-n_0(e)$ form a second distribution on F_1 and F_2 respectively for which the energy is I_0, the total amounts being μ, $-\nu$, the same as before. Writing $\sigma(e)=\mu_0(e)-\nu_0(e)$, $\sigma'(e)=m_0(e)-n_0(e)$, J_1', J_2' as the constant values of the new potential, we have

$$I(\sigma - \sigma') \geqq 0.$$

But

$$\begin{aligned} I(\sigma - \sigma') &= I(\sigma) + I(\sigma') - 2I(\sigma, \sigma') \\ &= J_1\mu - J_2\nu + J_1'\mu - J_2'\nu - \int w_0 d\sigma' - \int w_0' d\sigma, \end{aligned}$$

and

$$\int w_0 d\sigma' = \int_{F_1} w_0 d\mu_0' - \int_{F_2} w_0 d\nu_0' = (J_1 + \epsilon_1)\mu - (J_2 - \epsilon_2)\nu,$$

$$\int w_0' d\sigma = \int_{F_1} w_0' d\mu_0 - \int_{F_1} w_0' d\nu_0 = (J_1' + \epsilon_3)\mu - (J_2' - \epsilon_4)\nu,$$

where by the theorem stated above (for example, $w_0 \geqq J_1$ on F_1), we have $\epsilon_i \geqq 0$, $i=1, 2, 3, 4$. Hence,

$$I(\sigma - \sigma') = -\epsilon_1\mu - \epsilon_2\nu - \epsilon_3\mu - \epsilon_4\nu \leqq 0.$$

From this and the previous inequality it follows that $I(\sigma-\sigma')=0$, and, by Frostman's theorem, that $\sigma(e)\equiv\sigma'(e)$ for all sets e measurable (B). But then the potentials of the two mass distributions are also identical.

10. *Total Masses Variable.* We may consider various values of the total masses μ, ν. If $\nu=0$, $\mu=1$ we obtain the "equilibrium distribution" for a unit positive mass on F_1, as considered especially for $\alpha \geqq 1$ in the memoir cited. We notice however that there is also a unique equilibrium distribution for the case $0<\alpha<1$, under the same conditions. We denote the energy in this case by i_1.

Let $\mu=1$, $\nu=-1$ and write $i_0=J_1-J_2$. Let $\sigma(e)=\mu(e)-\nu(e)$, where $\mu(e)$, $-\nu(e)$ are distributions of total amounts m, $-m$ on F_1 and F_2 respectively, and define the averages

$$\bar{w}_1 = \frac{1}{m}\int_{F_1} w(P)d\mu(e_P), \qquad \bar{w}_2 = \frac{1}{m}\int_{F_2} w(P)d\nu(e_P).$$

If $\mu(e)$, $-\nu(e)$ are not identical with $m\mu_0(e)$, $-m\nu_0(e)$ respectively on all sets measurable (B), then

$$(14) \qquad I(\sigma) > m^2 i_0 > 0; \qquad \bar{w}_1 - \bar{w}_2 > m i_0 > 0.$$

In fact, $m^2 i_0 = I_0 < I(\sigma) = m(\bar{w}_1 - \bar{w}_2)$.

Finally, let us take μ, ν variable, but $\mu=\nu+1$ so that we shall consider the equilibrium distribution of a total algebraic mass of amount $\mu-\nu=1$, the positive mass $\mu(e)$ on F_1 and the negative mass $-\nu(e)$ on F_2. We show that under these conditions, *with μ, ν variable and $\mu=\nu+1$, there is an optimum distribution which makes I a minimum. Evidently in this case $J_1=J_2=I_0$, the minimum value.*

In fact, the reasoning of §8 of this Part III applies unless for every infinite subsequence of the $\mu_n(e)$, $\nu_n(e)$ the $\nu_n(F_2)$ fail to be bounded. In order to eliminate this possibility, it will be shown that I becomes indefinitely large with ν and hence cannot thus approach its lower bound.

Write $\mu(e)=p(e)+m(e)$ with $p(e)\geqq 0$, $m(e)\geqq 0$, $p(F_1)=1$, so that $m(F_1)=\nu(F_2)=\nu$. Let the potential of $p(e)$ be $u'(M)$ and that of $m(e)-\nu(e)$ be $w'(M)$. Then

$$I=\int_{F_1}[u'(M)+w'(M)]d[p(e_M)+m(e_M)]$$

$$-\int_{F_2}[u'(M)+w'(M)]d\nu(e_M)$$

$$=\left\{\int_{F_1}w'dm-\int_{F_2}w'd\nu\right\}+\int_{F_1}u'dp+\int_{F_1}u'dm-\int_{F_2}u'd\nu,$$

or

$$I\geqq \nu^2 i_0+i_1+0-\nu/\delta^\alpha,$$

δ being the minimum distance between points of F_1 and F_2. Accordingly I becomes infinite with ν.

11. *Special Case.* Nothing has been said about the more interesting and important problem in which δ may be zero, or in which F_1 and F_2 overlap. It is not certain that this general problem has a solution. The simplest possible problem of this kind is that of two concentric spherical surfaces, where the radius of the inner surface is variable. We shall see that a solution of the problem exists in this case. We take $\alpha<1$.

The formula for the potential of unit mass, spread uniformly on a spherical surface of radius r, at a point distant ξ from the center is found by integration to be the following:

$$(15)\quad u(r,\xi)=u(\xi,r)=\frac{1}{2(2-\alpha)r\xi}\left\{(r+\xi)^{2-\alpha}-|r-\xi|^{2-\alpha}\right\}.$$

Hence for a positive distribution of amount $1+\nu$ spread uniformly on a spherical surface of radius a and a negative one of amount $-\nu$ spread similarly on the concentric surface of radius b, $b<a$, the potential is

$$(16)\qquad V(\xi)=(1+\nu)u(a,\xi)-\nu u(b,\xi).$$

The equilibrium distribution for a given value of b, which we have already seen to exist, is given by the condition $V(a)=V(b)$, so that

$$\nu = \frac{u(b, a) - u(a, a)}{u(a, a) + u(b, b) - 2u(a, b)}.$$

The corresponding value of $I = (1+\nu)V(a) - \nu V(b) = V(a)$ is

$$I = u(a, a) - \nu\{u(b, a) - u(a, a)\},$$

or

$$I = u(a, a) - \frac{\{u(b, a) - u(a, a)\}^2}{u(a, a) + u(b, b) - 2u(a, b)}. \tag{16'}$$

We wish to show that the lower bound for I, as b varies, is taken on for some value of b in the open interval $0<b<a$. In fact, (16) and (16′) represent functions of b which are continuous and differentiable except at $b=0$ and $b=a$, and it is therefore merely necessary to show that b cannot approach 0 or a while I tends to its lower bound I_0.

Since $\alpha<1$, the potential of a positive mass distribution is strictly superharmonic. Hence $u(a, a)<u(a, \xi)$, $\xi<a$, that is, the potential of the uniform distribution of unit positive mass on the surface of radius a is greater inside than on the surface. Hence the energy can be reduced by putting a small negative mass uniformly on a sphere of radius b, $0<b<a$, and placing the equivalent positive mass on the sphere of radius a. Accordingly, for the energy as given by (16′) we have $I<u(a, a)$, and therefore

$$I_0 < u(a, a).$$

But now, if we let b tend to 0 in (16) and (16′), $u(b, b)$ becomes positively infinite, while the other quantities in the expressions remain finite. Hence

$$\lim_{b=0} \nu = 0; \qquad \lim_{b=0} I = u(a, a).$$

Similarly, if we let b tend to a, with $b<a$, we find by the elementary differentiation process that

$$\lim_{b=a-0} \nu = +\infty,$$

but again, that

$$\lim_{b=a-0} I = u(a, a).$$

Since $u(a, a) > I_0$, this lower bound I_0 will be attained for some value of b in the open interval $(0, a)$.

12. *The Principle of the Maximum.* Frostman limits his discussion in the main to functions $1/r^\alpha$, $\alpha > 1$, or to their generalization $\Phi(r)$ which retains the subharmonic property. In this case, for distributions of positive mass of total amount unity on a closed set F of positive mass, the equilibrium distribution corresponding to the minimum energy is constant on F except for a subset of zero capacity. This fact enables the author to extend the sweeping-out process of Poincaré and Gauss. In particular there is deduced the extension of Maria's theorem, established by him for $\alpha = 1$,* that the upper bound of the potential of an arbitrary distribution of mass is as great on the set where the mass lies as it is on the complementary set. On account of the strictly superharmonic character of the potential when $\alpha < 1$, this theorem does not hold when $\alpha < 1$.

From Maria's theorem follows the further fact that, if Q is a frontier point of F, P a variable point of F, and M of CF, then

$$\limsup_{M=Q} u(M) \leqq \limsup_{P=Q} u(P),$$

where $u(M)$ is the potential of an arbitrary distribution of positive mass on F. It therefore remains valid for $\alpha > 1$, but even this narrower statement appears not to be true for $\alpha < 1$.

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* A. J. Maria, *The potential of a positive mass and the weight function of Wiener*, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 485–489.

THE EXISTENCE OF ALGEBRAIC PLANE CURVES*

BY T. R. HOLLCROFT

I. Curves in General

1. *Introduction.* In this discussion, only proper or irreducible algebraic plane curves are to be dealt with; so, unless otherwise stated, the term "curve" will mean an irreducible, algebraic plane curve. The treatment will be from the point of view of algebraic geometry. General curves will be treated first, followed by a discussion of regular and irregular curves. Citations are given by number referring to a brief alphabetical bibliography at the end of the paper. All results that have been previously published are accompanied by a citation.

A curve is of order n, class m, genus p, with δ nodes, κ cusps, τ bitangents, ι statangents (contraction of "stationary tangents"). Julius Plücker [15], just a century ago, discovered a very remarkable set of relations involving the above seven characteristics. These relations, which may be expressed in various ways, are called "Plücker's equations." When any three characteristics are given, the other four are uniquely determined by these relations. A curve system, therefore, will be defined by given values of three characteristics. B. Segre [16] has chosen the three characteristics n, p, κ and represents a curve system by the symbol (n, p, κ). This notation will be used here.

The Plücker relations defining m, δ, ι, τ, in terms of n, p, κ are

$$m = 2(n + p - 1) - \kappa,$$
$$\delta = \tfrac{1}{2}(n - 1)(n - 2) - p - \kappa,$$
$$\iota = 3(n + 2p - 2) - 2\kappa,$$
$$\tau = \tfrac{1}{2}[(2n + 2p - \kappa)^2 - 4(5n + 7p - 6) + 11\kappa].$$

A curve system (n, p, κ) includes all curves with the given set of characteristics without regard to existence. Other systems may be obtained from a given system (n, p, κ) algebraically, that is, α nodes may be added by decreasing p by α; α cusps added by simultaneously increasing κ by α and decreasing p by

* An address delivered before the Society on February 20, 1937 in New York, by invitation of the Program Committee.

α; α nodes changed to cusps or α cusps to nodes by respectively increasing or decreasing κ by α. These statements hold for positive or negative values of α so that the term "added" implies algebraic addition.

A complete curve system (B. Segre [16], p. 32) is a system that is not contained in a larger system of curves with the same characteristics. A continuous system is one whose descriptive properties only are defined.

A curve system (n, p, κ) will be said to exist, that is, to define the characteristics of a curve that exists, when

(a) The algebraic equation of a curve of this system can be written in the polynomial form or in parametric form with one or more parameters and it can be proved that its polynomial equation cannot be factored, that is, is irreducible.

(b) A curve of this system can be constructed and proved to have an irreducible algebraic equation.

(c) A curve of this system can be transformed into a curve known to exist by any non-degenerate rational algebraic transformation.

Conditions (b) and (c) are really contained in (a) so that the question of the existence of a system (n, p, κ) becomes—does the curve system define a curve which has an irreducible algebraic equation?

The problem of the existence of plane curves is of basic importance because the existence of curves and manifolds of higher space may be made to depend upon that of plane curves.

In the plane itself, the only noteworthy progress made has been in the case of curves whose only singularities are distinct nodes and cusps. Snyder [20] in 1908 and Severi [19] in 1921 proved that curves with no cusps and any number of nodes up to and including the maximum exist. For curves with only nodes and cusps, the problem of existence is closely related to that of determining the maximum number of cusps of a curve.

2. *Conditions for Existence.* Conditions for the existence of a curve system (n, p, κ) are necessary, sufficient, or both necessary and sufficient.

(1) *Necessary conditions.* The most obvious necessary conditions for the existence of curves are Plücker's equations. From these relations are derived formulas for the maximum number

of cusps (Clebsch [2]; Lefschetz [13]; Hollcroft [8]) of a curve of given order and genus. Such limits, called Plücker limits, are necessary conditions.

Cramer [4] in 1750 found certain necessary conditions for the existence of curves with multiple points from the maximum number of intersections of such a curve with another curve through these multiple points. He made a table of the possible multiple points for curves of order $n \leqq 8$. Using a similar process, Plücker ([15], p. 215) derived the well known necessary condition $\delta+\kappa \leqq \frac{1}{2}(n-1)(n-2)$.

If the reality of the singularities of the curve is taken into consideration, Klein's Theorem [12], $n+i+2t''=m+k+2d''$, wherein (i) $[k]$ is the number of real (statangents) [cusps] and (t'') $[d'']$ the number of isolated (bitangents) [nodes], is an additional necessary condition for the existence of the curve. The limits to the number of real cusps (Hollcroft [10], pp. 777, 781) for a curve of given order and genus, since they are derived from this theorem and Plücker's equations, are, therefore, necessary conditions.

(2) *Sufficient conditions.* There are few sufficient conditions for existence that are not also necessary. Sufficient conditions only, however, are given by the two following theorems due to B. Segre ([16], pp. 36–37):

THEOREM 1°. *If a continuous regular system* (n, p, κ) *exists, there exist also the regular systems* (n, p_1, κ_1) *where* p_1 *and* κ_1 *satisfy the inequalities* $0 \leqq \kappa_1 \leqq \kappa$, $p \leqq p_1 \leqq \frac{1}{2}(n-1)(n-2)-\kappa_1$.

THEOREM 2°. *Given two regular systems* (n_1, p_1, κ_1) *and* (n_2, p_2, κ_2), *if there exists a curve of one system that touches a curve of the other in* h *simple points, and the additional intersections* n_1n_2-2h *are distinct points, then there exists a regular system* (n, p, κ) *such that*

$$n = n_1 + n_2, \qquad p = p_1 + p_2 + h - 1, \qquad \kappa = \kappa_1 + \kappa_2 + h.$$

Because of the paucity of sufficient conditions for the existence of curves, these theorems are of great importance. The latter theorem has given added significance to the problem of determining the maximum number of contacts of two curves of given systems.

(3) *Necessary and sufficient conditions.* Conditions that are both necessary and sufficient for the existence of curves are implied in the definition of existence in the preceding section, namely, that an irreducible polynomial equation of the curve can be either obtained or proved to exist. The fact that the equation is algebraic, rational, integral, and irreducible is sufficient for the curve to satisfy all necessary conditions.

3. *The Dimension of a Curve System.* The equation of the general curve of order n contains $\frac{1}{2}n(n+3)$ independent coefficients. The curves of this system are in (1, 1) correspondence with the points of $S_{n(n+3)/2}$ and the system is said to be of dimension $n(n+3)/2$. The values of a certain number of coefficients may be determined in terms of the others by algebraic relations associated with geometric conditions on the curve. Such a condition is expressed by a system of $\alpha \geqq 1$ algebraic equations in the coefficients. The $\frac{1}{2}(n+1)(n+2)$ homogeneous coefficients of the general system of curves may be considered the homogeneous coordinates in $S_{n(n+3)/2}$. The system of α independent equations in the coefficients that are associated with a certain condition thus define uniquely an algebraic manifold in $S_{n(n+3)/2}$ whose dimension is $n(n+3)/2-\alpha$ and whose order is the product of the degrees of the α equations of the system. (Severi [19]).

Conditions involving relations among the coefficients of a curve are of two kinds; first, metric conditions, that is, those associated with position in the plane. Metric relations may be linear or of any order in the coefficients. The second kind of condition is that defining a property of the curve which is invariant under a plane collineation. The associated algebraic relations among the coefficients are called invariants of the curve. The order of an invariant is always greater than unity. Lefschetz ([13], p. 24) has proved that a curve of order n and genus $p \geqq 2$ cannot have more than $\frac{1}{2}n(n+3)-8$ independent invariants.

With each node of a curve is associated one invariant and with each cusp, two invariants. It has been proved (Severi [19]; Snyder [20]) that the δ nodal invariants associated with any number $\delta \leqq \frac{1}{2}(n-1)(n-2)$ of distinct nodes of a curve of order n are independent. It is also true that the cuspidal invariants are all independent for curves of low order that have been treated in

detail algebraically. Likewise, from Plücker's equations there results the necessary relation

$$d_0' = \tfrac{1}{2}n(n+3) - \delta - 2\kappa = \tfrac{1}{2}m(m+3) - \tau - 2\iota,$$

which (for the curve system whose only singularities are distinct nodes and cusps) *appears* to express the well known fact that the number of independent coefficients in the equation of a curve in point coordinates equals the number in the equation of the same curve in line coordinates.

In 1913, Lefschetz ([13], p. 26) stated his postulate of singularities which is to the effect that each distinct cusp of a curve is always associated with two independent invariants. This postulate is a statement of what had been found true in all cases investigated up to that time. It could not be proved true in all cases, but the above relation for d_0' seemed to indicate its validity.

In 1929, B. Segre ([16], pp. 34–35; [17]) discovered a curve system with distinct cusps, to be discussed in section 8, that does not satisfy the Lefschetz postulate. This contradiction is sufficient to prove that the postulate does not hold in all cases.

With B. Segre [16], the first virtual dimension of the system (n, p, κ) is $d_0' = 3n+p-\kappa-1$. The virtual dimension d_0 of a curve system with given singularities (not defined in position) is the number of independent coefficients that would remain in the equation of a curve of this system if all of the singularity invariants were considered independent. The effective dimension d is the actual number of independent coefficients in the equation of such a curve system. For a curve system whose only singularities are distinct nodes and cusps, $d_0' = d_0$.

The effective dimension d of an irreducible curve system is also defined as the dimension, reduced by one, of the characteristic series of the curve. The characteristic series has an index of specialty $i \geqq 0$ and a deficiency $\omega \geqq 0$. By the Riemann-Roch Theorem, $d+\omega = d_0+i$. B. Segre ([16], p. 33) has proved that $d \geqq d_0$ for all complete, continuous curve systems. He calls a curve system *regular* for which $d = d_0$ and *irregular* for which $d > d_0$.

Although proved not to apply in all cases, the Lefschetz postulate can now be partially revived, that is, all regular curves satisfy the Lefschetz postulate. The converse, however, is

not true since other than cuspidal invariants may be dependent (see §10).

4. *Curves with Distinct Nodes and Cusps.* In 1913, Lefschetz [13] derived from Plücker's equations the following limits to the number of cusps of a curve of given order and genus:

For
$$0 \leqq p \leqq p_0,$$
$$\kappa \leqq \tfrac{3}{2}(n + 2p - 2);$$
$$p_0 < p \leqq \tfrac{1}{6}(n - 2)(n - 3),$$
$$\kappa \leqq 2(n + p) - \tfrac{1}{2}\{11 + (24p - 8n + 25)^{1/2}\};$$
$$\tfrac{1}{6}(n - 2)(n - 3) \leqq p \leqq \tfrac{1}{2}(n - 1)(n - 2),$$
$$\kappa = \tfrac{1}{2}(n - 1)(n - 2) - p.$$

In the above formulas, $p_0 = [\frac{1}{2}\{n+1-(4n+1)^{1/2}\}]$. The genus p_0 is the least genus of the curve systems of minimum class for a given n

The Plücker limits for $p \leqq p_0$ and $p > p_0$ result from the inequalities $\iota \geqq 0$ and $\tau \geqq 0$ respectively, that is, the fact that none of Plücker's numbers can be negative when the curve is irreducible. Curves with the maximum number of cusps for a given order and genus have been designated maximal cuspidal curves (Hollcroft [9]).

Lefschetz [13] and Coolidge [3] have established the existence of maximal cuspidal curves of genus $p \leqq p_0$ and by Segre's Theorem 1° ([16], p. 36), the existence of all curves with distinct nodes and cusps of genus $p \leqq p_0$ is proved. All such curves are also regular ([16], p. 33).

If a curve exists, its reciprocal exists. The reciprocals of curves of order n and genus $p \leqq p_0$ when $n-2p+2 < \kappa \leqq \frac{3}{2}(n+2p-2)$ are curves of order m for which it may be that $p > [\frac{1}{2}\{m+1-(4m+1)^{1/2}\}]$. In such cases, however, the number of cusps ι of the reciprocal is always less than $m+2p-2$. The existence of these curves of genus greater than p_0 does not aid in establishing the existence of curves of genus greater than p_0 with a number of cusps equal to or near the Plücker limit for that genus. With the exception of an irregular system shown to exist in §8, no general system of maximal cuspidal curves of genus $p > p_0$ has been proved to exist.

The two following theorems define curve systems whose reciprocals are maximal cuspidal curve systems, and conversely:

1. *Every curve system of order n for which $\kappa \leqq 1$ and $\delta \geqq n-4$ is the reciprocal of a maximal cuspidal curve system of genus $p \leqq p_0$.*

2. *Every curve system of order n and class m with $\kappa \geqq 0$ cusps and $\delta \leqq n-5$ nodes is the reciprocal of a maximal cuspidal curve system of order m and genus p such that $p_0 < p \leqq (m-2)(m-3)/6$.*

II. Regular Curve Systems

5. *B. Segre's Results.* By establishing the completeness of the characteristic series* of a complete continuous system of curves (n, p, κ) for which $\kappa < 3n$ and n is sufficiently large,† B. Segre ([16], p. 33) proved:

All continuous systems of irreducible algebraic curves whose only singularities are distinct nodes and cusps, provided that $\kappa < 3n$, are regular, that is for such a system, $d_0' = d_0 = d = 3n + p - 1 - \kappa$.

In the same paper ([16], pp. 36–37), he gave the two Theorems 1^0 and 2^0, which have been stated as sufficient conditions for existence in §2. A third theorem, Theorem 3^0 ([16], p. 37) is derived from Theorem 2^0 by considering the second curve system as a line counted twice, that is, $(n_2, p_2, \kappa_2) \equiv (2, -1, 1)$.

Theorem 3^0. *If a regular system (n_1, p_1, κ_1) exists for $n_1 \geqq 3$, there exists also the regular system (n, p, κ) such that*

$$n = n_1 + 2, \qquad p = p_1 + n_1 - 2, \qquad \kappa = \kappa_1 + n_1 + 1.$$

From the four preceding theorems, Segre derives the existence theorem:

There exist continuous regular systems $(2n, n^2-3n+1, n^2)$ for $n \geqq 3$ and $(2n+1, n^2-2n+1, n^2+n-1)$ for $n \geqq 1$ of irreducible curves with only cusps.

Theorems 1^0, 2^0, 3^0 are given without proof.

There are questions as to the field of validity of Theorem 3^0

* The proof is similar to that of Severi [19] in the case of nodes. Severi proved that all continuous systems $(n, p, 0)$ are regular.

† By Plücker's equations, the least value of n for which $\kappa = 3n$ is 11. This is a self dual curve and probably does not exist. For $m \neq n$, when $\kappa = 3n$, $m = n(n-10)$ and $p - 1 = (m+n)/2$. The system (12, 19, 36) is the one of lowest order for $m \neq n$. The least value of n for which this theorem holds when $\kappa = 3n-1$ probably lies between 12 and 17.

and therefore of the above existence theorem derived partially from this theorem. This will be discussed in section 7.

6. *Zariski's Results.* Zariski studied the surface $z^n=f(x, y)$ where $f(x, y)=0$, the branch curve [21] of the surface, is an algebraic curve with given singularities. In 1929, he proved the theorem [22]:

If $n=q^\alpha$, where q is a prime number, and $f(x, y)=0$ is an irreducible algebraic curve, then the surface $z^n=f(x, y)$ is regular.

A regular surface F of order n is one for which $p_g-p_a=0$, where p_a, the arithmetic genus of F, is one more than the virtual dimension of the system of adjoint surfaces of order $n-4$ and p_g, the geometric genus of F, is one more than the effective dimension of the same adjoint system. The geometric genus is also defined as the number of everywhere finite abelian double integrals of the first kind associated with the surface F. The difference $I=p_g-p_a\geqq 0$ is called the irregularity of F.

Let $f(x, y)=0$ be a curve f of order n with no singularities other than distinct nodes and cusps. The surface F with equation $z^n=f(x, y)$ has compound double points at the nodes and cusps of f in the plane $z=0$. An adjoint surface $\phi(x, y, z)=0$ of order $n-4$ does not pass through the ordinary double points of F, but may pass through compound double points of F.

By studying the behavior, at these double points of F, of the abelian integrals $\int\int[\phi(x, y, z)/z^{n-1}]dx\,dy$, Zariski ([24], p. 488) proved these theorems:

1. The adjoint surfaces of F do not pass through double points of F which are nodes of f.

2. At each double point of F which is a cusp of f, the adjoint surfaces of F have $[n/6]$ consecutive basis points ($[n/6]$ is the largest integer contained in the quotient $n/6$).

The second result leads to the following limit ([24], p. 500) to the number of cusps of an irreducible curve of order n,

$$\kappa < \tfrac{1}{2}(n-\beta)(n-\beta-3)+2, \qquad \beta=[(n-1)/6].^*$$

For a given n, this limit defines a smaller number of cusps than the Plücker limit in certain cases ([24], p. 501).

* In reference [25], p. 174, the equality sign is also given in this limit. In accordance with the derivation, the equality sign is not admissible.

7. *The Existence of Curves of Genus* $p \leqq 5$. Lefschetz ([13], pp. 37–39) investigated the existence of curve systems of low genus with only distinct nodes and cusps and proved that all maximal cuspidal curve systems of genus $p \leqq 5$ exist with the three exceptions (7, 3, 11), (8, 4, 14), (8, 5, 16).

As stated in §4, all curves having only distinct nodes and cusps and of genus $p \leqq p_0$ exist. In determining the existence of curve systems of genus $p \leqq 5$, therefore, it is only necessary to deal with curve systems for which $p_0 < 5$. The inequality $p_0 < 5$ gives $n \leqq 16$. The reciprocals of all such curves of order $n \geqq 10$ are curves of order $n \leqq 8$ which do not have the maximum number of cusps for that order and genus, so that all exist. For $n = 9$, however, among the reciprocals there are two additional systems, (7, 4, 11) and (8, 5, 15), which cannot be proved to exist.*

Zariski ([24], p. 501) has shown that the systems (7, 4, 11) and (7, 3, 11) do not exist. This contradicts B. Segre's Theorem 3^0. The curve system of order 7 with 11 cusps is the curve of lowest order satisfying the Plücker limits that does not exist.†

In a separate paper [23], Zariski gives a direct proof by plane geometry that the system (8, 5, 16) does not exist. He points out in a footnote in his paper ([23], p. 309) that B. Segre has stated ([16], p. 38, footnote) that the system (8, 5, 16) exists as a result of the preceding theorems of that paper.

A contradiction between Zariski's limit $\kappa < \frac{1}{2}(n-\beta)(n-\beta-3) + 2$, $\beta = [(n-1)/6]$, and B. Segre's Theorem 3^0 occurs only in the case of curves of orders 7 and 8. B. Segre's existence theorem which affirms the existence of regular curve systems $(2n, n^2-3n+1, n^2)$, $n \geqq 3$, and $(2n+1, n^2-2n+1, n^2+n-1)$, $n \geqq 1$, coincides with the Plücker limit for $n \leqq 8$ and gives a much smaller limit for $n \geqq 9$. The Zariski limit defines a maximum number of cusps smaller than the Plücker limit only for the six orders $n = 7, 8, 13, 14, 19, 25$. For all orders except 7 and 8 (in which cases B. Segre's existence theorem allows 11 and 16 cusps, Zariski's limit 10 and 15 respectively), the existence state-

* The three exceptional systems given by Lefschetz constitute all of the doubtful self dual maximal cuspidal curve systems for $p \leqq 5$. Of the two additional systems, (7, 4, 11) is of higher genus than $(n-2)(n-3)/6$ for $n = 7$ and (8, 5, 15) is "between" the two doubtful systems of order 8 given by Lefschetz.

† The conclusion of Crone [5] that the system (6, 2, 7) does not exist is not justifiable.

ment of B. Segre defines a smaller number of cusps than the Zariski limit.

In addition to showing that the curve systems (7, 4, 11), (7, 3, 11) and (8, 5, 16) do not exist, Zariski's theorem ([**22**], p. 494) limits the positions of cusps as follows on certain curve systems:

(7, 5, 10), the ten cusps cannot lie on a cubic;

(8, 9, 12), the twelve cusps cannot lie at the intersections of a cubic and a quartic;

(8, 6, 15), the 15 cusps cannot lie on a quartic;

(9, 7, 21), the 21 cusps cannot lie on a quintic.

The first two of these cases are mentioned by Zariski ([**24**], p. 502). The existence of the last two curve systems is in doubt, but their existence is permitted by the theorem which restricts the positions of their cusps.

The existence of a curve of order 9 with 18 cusps will now be established.

The Cayleyan of an elliptic cubic is a sextic with 9 cusps which has nine contacts with the cubic. Therefore the regular systems (6, 1, 9) and (3, 1, 0) contain curves that have 9 contacts. Applying Segre's Theorem 2^0 (see §2), we find that set $(n_1, p_1, \kappa_1) \equiv (6, 1, 9)$, $(n_2, p_2, \kappa_2) \equiv (3, 1, 0)$, and $h = 9$. The theorem then asserts the existence of the regular curve system (9, 10, 18).

The reciprocal of (7, 4, 11) is (9, 4, 17), which does not exist since (7, 4, 11) does not exist. In the above paragraph, the system (9, 10, 18) has been shown to exist, which implies that all curve systems $(9, p, \kappa)$ exist for $p \geqq 10$ and $\kappa \leqq 18$.

This is the first time that the existence has been established of a curve with a number of cusps greater than that possessed by a curve of the same order that does not exist. This adds a new conception to the problem of existence—that distinct nodes added to the singularities of an existing curve system with only distinct nodes and cusps, within the Plücker limits, may give a system that does not exist. In this case, the system (9, 10, 17) exists. Six nodes added to this system give (9, 4, 17) whose reciprocal, by Zariski's theorem, does not exist.

It is entirely possible, from the following point of view, for the system (9, 4, 17) to be non-existent while the system (9, 10, 18) exists. Since the system (9, 4, 17) does not exist because (and only because, so far as known proofs are concerned) the

system (7, 4, 11) has too many cusps for its order, then, reciprocally, the system (9, 4, 17) does not exist *because it has too many statangents for its class*. This does not imply that the system (9, 4, 17) has too many cusps for its order, but that the *combination* of 17 cusps and 7 nodes is too much, in that it defines a reciprocal system with too many cusps for its order.

III. Irregular Curve Systems

8. *The Curve System f*. The curve system of order 6α

$$f \equiv \phi_{3\alpha}^{2} + \lambda\psi_{2\alpha}^{3} = 0,$$

in which $\phi_{3\alpha}=0$, $\psi_{2\alpha}=0$ are two entirely general curve systems of orders 3α and 2α respectively, has been long known, and it has also been known that this system has $6\alpha^2$ cusps at the intersections of $\phi_{3\alpha}$ and $\psi_{2\alpha}$. The new property of this system discovered by B. Segre ([16], p. 34) is its irregularity for $\alpha \geqq 3$.

The equation of f contains

$$\tfrac{1}{2}(3\alpha + 1)(3\alpha + 2) + (\alpha + 1)(2\alpha + 1) - 1 = \tfrac{1}{2}(\alpha + 1)(13\alpha + 2)$$

non-homogeneous coefficients. The proof of the irregularity of f consists in showing that all of these coefficients are independent. Segre proves this by showing that the system f is a complete, continuous, irreducible system of curves with a complete characteristic series.

The independence of the coefficients is also proved algebraically. Of the $6\alpha^2$ intersections of $\phi_{3\alpha}$ and $\psi_{2\alpha}$, only $6\alpha^2-(2\alpha-1)(2\alpha-2)/2=(\alpha+1)(4\alpha-1)$ points chosen on $\phi_{2\alpha}$ count as independent conditions in the determination of $\phi_{3\alpha}$. Of these $(\alpha+1)(4\alpha-1)$ points, assume that $\alpha(2\alpha+3)$ are chosen arbitrarily, thus determining $\psi_{2\alpha}$, and that the remaining $2\alpha^2-1$ are chosen on $\psi_{2\alpha}$. Then $\phi_{3\alpha}$ still contains $3\alpha(3\alpha+3)/2 -(\alpha+1)(4\alpha-1)=\frac{1}{2}(\alpha+1)(\alpha+2)$ arbitrary parameters and therefore f contains $\frac{1}{2}(\alpha+1)(\alpha+2)+1$ arbitrary parameters. The position of a cusp in the plane is determined by two parameters, on a given curve by one parameter. Of the $(\alpha+1)(4\alpha-1)$ given cusps on $\psi_{2\alpha}$, only $\alpha(2\alpha+3)$ positions are independent and account for two parameters each, while the remaining $2\alpha^2-1$ positions on $\psi_{2\alpha}$ account for one parameter each. Then, so far as determining the *positions* of $(\alpha+1)(4\alpha-1)$ cusps is concerned, the number of conditions on f would be $2\alpha(2\alpha+3)$

$+2\alpha^2-1=6\alpha^2+6\alpha-1$. If the positions of these $(\alpha+1)(4\alpha-1)$ cusps are considered as *not* given, there would be restored to the system f, on which they were considered as given, $6\alpha^2+6\alpha-1$ parameters. It was shown that the system f for which the positions of these cusps were given contained $\frac{1}{2}(\alpha+1)(\alpha+2)+1$ parameters. Therefore a curve system f for which no positions of the $6\alpha^2$ cusps are given contains

$$\tfrac{1}{2}(\alpha+1)(\alpha+2)+1+6\alpha^2+6\alpha-1=\tfrac{1}{2}(\alpha+1)(13\alpha+2)$$

independent parameters.

In the above proof, it was revealed that as to the *positions* of the $6\alpha^2$ cusps of f: $\alpha(2\alpha+3)$ are independent and require two parameters each; $2\alpha^2-1$ are on $\psi_{2\alpha}$ and require one parameter each; $(\alpha-1)(2\alpha-1)$ are gratuitous, that is, they follow from the above.

The characteristics of f are these: $n=6\alpha$, $m=6\alpha(3\alpha-1)$, $p=12\alpha^2-9\alpha+1$, $\delta=0$, $\kappa=6\alpha^2$, $\iota=12\alpha(5\alpha-3)$, $\tau=27\alpha(3\alpha-2)\cdot(2\alpha^2-1)$, $d_0=3\alpha(2\alpha+3)$, $d=(\alpha+1)(13\alpha+2)/2$, $I=d-d_0=(\alpha-1)(\alpha-2)/2$.

As will be seen from the characteristics of f, the reciprocal f' of f has $\tau=0$ which is a condition, when $p>p_0$, for the maximum number of cusps. The reciprocal f' is of order $6\alpha(3\alpha-1)$ and genus $12\alpha^2-9\alpha+1$. For this order, the value of p_0 is always less than the genus of f' for all values of α. Therefore this curve system f has still another new property—its reciprocal is the first maximal cuspidal curve system of general order known to exist for $p>p_0$.

Since the system f has no invariants other than cuspidal invariants and since for f we have $I=(\alpha-1)(\alpha-2)/2$, it follows that out of the total of $12\alpha^2$ cuspidal invariants, I are dependent. This does not mean, necessarily, that some of the cusps of f are dependent. Since a cusp is the limiting case of a node, the two cuspidal invariants take the place of the nodal invariant. Since all nodal invariants have been proved independent, it is probable that at least one of the invariants associated with any cusp is independent.

Since in the case of a curve with distinct nodes and cusps, $d_0=n(n+3)/2-\delta-2\kappa=m(m+3)/2-\tau-2\iota$, and since the effective dimension d of a curve considered as a locus is equal to that of the same curve considered as an envelope, $I=d-d_0$ has the same value for the given curve considered either as a locus or as

an envelope. Nodal invariants are independent, hence, reciprocally, bitangent invariants of a class curve are independent. Therefore if a curve, considered as a locus, has a certain number I of dependent cuspidal invariants, the same curve considered as an envelope has I dependent statangent invariants.

Therefore the reciprocal of an irregular system of irregularity I is an irregular system of irregularity I, and if the original system has K dependent cuspidal invariants, the reciprocal system has K dependent cuspidal invariants.

Hence the reciprocal of f, which has been shown to be a maximal cuspidal curve system f' of genus $p > p_0$, has $(\alpha-1)(\alpha-2)/2$ dependent cuspidal invariants.

The system f is irregular because its cusps occur at the intersections of curves of lower order and not because of an excessive number of cusps for the order of f. It is highly probable that systems of order 6α with $6\alpha^2$ cusps, unrestricted a priori in position, exist. In the discussion below, this is shown to be true in the case $\alpha = 1$.

Zariski ([24], p. 503) has proved the following theorem:

If $f(x, y) = 0$ is an irreducible curve of order n possessing nodes and cusps only, a necessary and sufficient condition that the surface $z^n = f(x, y)$ be irregular is that n be divisible by six and that the system $\left| C_{(5n-18)/6} \right|$ of the curves of order $(5n-18)/6$ passing through the cusps of the curve f should be superabundant. If these conditions are satisfied, the irregularity of the surface is equal to the superabundance of the above system $\left| C_{(5n-18)/6} \right|$.

This theorem requires that all surfaces $z^n = f(x, y)$ be regular if $f(x, y) = 0$ is an irreducible curve of order n with only distinct nodes and cusps whose order n is not divisible by six. This is much more general than the former theorem which required n to be the power of a prime in order for $z^n = f(x, y)$ to be regular.

The system $f(x, y) \equiv \phi_{3\alpha}^2 + \lambda\psi_{2\alpha}^3 = 0$, under this theorem, is the only irreducible system known for which the surface $z^n = f(x, y)$ is irregular.

The special case of f for $\alpha = 1$ defines a system of sextics C_6 with six cusps on a conic. This system C_6 is of dimension $d = d_0 = 15$, the same as the dimension of a sextic C_6' with six cusps in general position.* Zariski ([21], p. 320) has proved

* When the positions of the cusps are given, however, the dimensions of these two sextic curve systems are not equal. The C_6 with 6 cusps at given

that C_6 is the branch curve of a general cubic surface and that the fundamental groups of the two types of sextics are distinct. B. Segre ([17], p. 559) has shown that a C_6' exists. The fact that C_6 and C_6' belong to distinct systems contradicts a theorem by Albanese ([1], part IV).

The above example gives rise to another discovery in curve theory—that a system of irreducible curves with given characteristics (n, p, κ) does not necessarily form one irreducible complete system (Zariski [25], p. 175).

The system f is of the form $\phi_a^2+\psi_b^3=0$, $2a=3b$. What are the properties of the system defined by this equation when $2a\neq 3b$?

In order to study this question, consider the continuous system

$$f_2 \equiv \phi_{3\alpha+1}^2 + \lambda\psi_{2\alpha}^3 = 0.$$

This system is of order $6\alpha+2$ and has $2\alpha(3\alpha+1)$ cusps at the intersections of $\phi_{3\alpha+1}$ and $\psi_{2\alpha}$. All of the $(13\alpha^2+21\alpha+6)/2$ coefficients can be proved independent. Also $d_0=6\alpha^2+17\alpha+5$; hence $d-d_0=\frac{1}{2}(\alpha^2-13\alpha-4)$. For $\alpha\leqq 13$, the system f_2 is incomplete. For $\alpha\geqq 14$, the system f_2, *if irreducible*, is a complete, continuous, irregular system.

The equations f_3 and f_4 of systems of orders $6\alpha+3$ and $6\alpha+4$ respectively may be written similarly and the systems have similar properties.

The surfaces $z^{6\alpha+i}=f_i$, $(i=2, 3, 4)$, however, are found to be irregular for all values of α for which the above curve systems would be irregular. Since the orders of these surfaces are not multiples of six, by Zariski's theorem ([24], p. 503), quoted in this section, all of these curve systems must be reducible for the values of α for which they would be irregular.

9. *The Plane Sections of Tangent Cones.* The sections by a plane π of the tangent cones to an algebraic surface of order ν form a complete, irreducible system of curves of order $\nu(\nu-1)$

points on a conic is an ∞^4 system and the C_6', the positions of whose cusps are independently assigned, is an ∞^3 system.

For any α, if the positions of all the $6\alpha^2$ cusps are given, the system f is of dimension $1+(\alpha+1)(\alpha+2)/2$. If the positions of the $6\alpha^2$ cusps of a curve system of order 6α, $\alpha\geqq 2$, are considered independent of each other, the positions of only $3\alpha(2\alpha+3)/2$ cusps can be chosen arbitrarily and the positions of the remaining $3\alpha(2\alpha-3)/2$ cusps follow from these.

with $\kappa=\nu(\nu-1)(\nu-2)$ and $\delta=\frac{1}{2}\nu(\nu-1)(\nu-2)(\nu-3)$. These are branch curves of the surface. The parametric equations of such a system may be written $\phi(x, y, z)=0$, $\partial\phi/\partial z=0$, wherein z is the parameter and $\phi=0$ the equation of the surface. B. Segre [13] has proved that, for this system, $d-d_0=(\nu-2)(\nu-3)(\nu-4)/6$. The system is thus irregular for $\nu\geqq 5$. B. Segre has also shown that the nodes and cusps of this system lie at the intersections of curves of lower order.

10. *An Irregular System of Quartic Curves.* In the two preceding sections, two well known curve systems have revealed a new property, that of irregularity. In this section, another well known curve system, the quartics with three biflecnodes, will be shown to be irregular.

A rational trinodal quartic that has biflecnodes at two of the nodes must have a biflecnode at the third node. The equation of the tri-biflecnodal quartic may be written in the form

$$x_1^{-2}+x_2^{-2}=x_3^{-2}. \quad \text{(Hilton [7], p. 285.)}$$

Making the sides of the coordinate triangle any three lines of the plane and writing the equation in integral form, we see that

$$q\equiv(bx)^2(cx)^2+(ax)^2(cx)^2-(ax)^2(bx)^2=0,$$

wherein $(ax)\equiv a_1x_1+a_2x_2+a_3x_3, \cdots$. The equation q contains 7 independent coefficients, that is, for the system q, the effective dimension d is equal to 7. The system q is continuous and complete.

Each node is associated with one invariant. Distinct statangents account for no invariants on an order curve, but if both nodal tangents coincide respectively with two statangents with their points of contact at the node, the singularity is associated with three invariants (Hollcroft [11], p. 265). The system q with three biflecnodes is therefore of virtual dimension $d_0=5$.

Therefore $d-d_0=2$ and the system q is irregular. The order four is evidently the lowest possible order for a complete, continuous, irregular curve system.

The reciprocal q' of q is a rational maximal cuspidal sextic with the six cusps on a conic and so associated in three pairs that each pair has a common cuspidal tangent. These three common tangents form a self conjugate triangle of the conic on which lie the six cusps.

The equation of q' contains the same number of independent constants as that of q, that is, $d=7$. The invariant postulation of two cusps with a common tangent is six, two for each cusp and two for the coincident cuspidal tangents. This coincidence implies that four tangents coincide, but since there are two distinct points of contact, the tangent is still a bitangent. The system q' has four nodes. For q', therefore, $d_0=27-18-4=5$. Hence $d-d_0=2$.

The most interesting thing about this system q and its reciprocal q' is that they exhibit cases of irregular systems in which cuspidal invariants are not the dependent invariants. A property of the reciprocal of the tri-biflecnodal quartic is stated as follows: If a rational maximal cuspidal sextic has two pairs of cusps so situated that the cuspidal tangents of both pairs coincide, the cuspidal tangents of the third pair must coincide. The sextic is considered as having the cusps before these further conditions are applied. Thus in the case of q', the two dependent invariants are associated with the coincidence of the third pair of cuspidal tangents. In the case of q, the two dependent invariants are associated with the coincidence, respectively, of two statangents with the two nodal tangents at the third node.

In both cases, the dependent invariants are associated with the coincidence of tangents, but not with the coincidence of nodal tangents to form a cusp.

11. *Curve Systems with Negative Virtual Dimension.* It is unlikely that curves exist with only distinct nodes and cusps for which d_0 is negative, although this is permitted by Plücker's equations. However, systems with higher singularities may exist for which $d_0<0$.

The irregular system f is a special case of a general system of the form $\phi_{b\alpha}^{a}+\lambda\psi_{a\alpha}^{b}=0$, $a>b$. This system is of order $ab\alpha$ with $ab\alpha^2$ compound b-fold points, each with b coincident tangents. The curve system is irregular and may be reducible for certain values of a, b, α. Its effective dimension is easily found, but the virtual dimension is difficult to obtain in a general case because each multiple point contains latent consecutive multiple points.

If $a=b+1$, the b-fold points are simple multiple points with b coincident tangents. The system f_b is of order $\alpha b(b+1)$ and has $\alpha^2 b(b+1)$ simple multiple points of order b at the intersections of $\phi_{ab}=0$ and $\psi_{a(b+1)}=0$. Each b-fold point contains

(penultimately) $b-1$ cusps and $(b-1)(b-2)/2$ nodes and is associated with $(b^2+3b-6)/2$ invariants. Hence

$$d_0 = \alpha b(b+1)(6\alpha - 2\alpha b + 3)/2.$$

The coefficients of f_b are independent, so that for f_b

$$d - d_0 = \tfrac{1}{2}\alpha b[(b+1)(2\alpha b - 6\alpha - 3) + (\alpha b + \alpha + 3)] + \tfrac{1}{2}(\alpha+1)(\alpha+2).$$

Setting $d_0<0$ and solving for b, we see that $d_0<0$ for $\alpha=1$, $b\geqq 5$, and for $\alpha\geqq 2$, $b\geqq 4$.

The curve system of lowest order of this form for which d_0 is negative is of order 30 ($\alpha=1$, $b=5$) and has the equation

$$\gamma \equiv \phi_5^6 + \lambda\psi_6^5 = 0.$$

The system γ has 30 simple 5-fold points, each with 5 coincident tangents and is of genus 135. Each 5-fold point is associated with 17 invariants. For γ, therefore, $d=48$, $d_0=495-510=-15$, and $d-d_0=48-(-15)=63$.

For $\alpha=2$, $b=4$, the curve system is

$$\mu \equiv \phi_8^5 + \lambda\psi_{10}^4 = 0.$$

The system μ is of order 40 and has 80 quadruple points, each with four coincident tangents. For this system, $d=110$, $d_0=860-880=-20$ and $d-d_0=130$.

It may be that equations of this form do not always represent irreducible systems. The above systems γ and μ, however, satisfy the known necessary conditions for existence, possess only distinct, simple multiple points, and have a negative virtual dimension.

An interesting question is raised by systems of this kind, as to whether the $d-d_0$ dependent invariants occur entirely among the cuspidal invariants implied in the multiple points (of which each b-fold point has $b-1$), or entirely among the combination invariants (of which each simple multiple point of order b has $b-2$), or among both of these sets of invariants. In the examples of irregular curve systems given in the preceding sections, the dependent invariants have always been associated with the coincidence of tangents, and with the coincidence of nodal tangents to form a cusp except in the case of the irregular system of quartics discussed in the preceding section.

BIBLIOGRAPHY

1. Albanese, G., *Sui sistemi continui di curve piane algebriche*, Pisa, Nistri, 1923.

2. Clebsch, A., Journal für Mathematik, vol. 64 (1864), p. 51.

3. Coolidge, J. L., *On the existence of curves with assigned singularities*, this Bulletin, vol. 28 (1922), pp. 451–455.

4. Cramer, G., *Introduction à l'Analyse des Lignes Courbes Algébriques*, Geneva, 1750, pp. 455–459.

5. Crone, C., *Sur une espèce de courbes symmetriques de la sixième classe*, Acta Mathematica, vol. 2 (1883), pp. 81–96.

6. Haskell, M. W., *The maximum number of cusps of an algebraic plane curve and enumeration of self dual curves*, this Bulletin, vol. 23 (1917), pp. 164–165.

7. Hilton, H., *Plane algebraic curves*, Oxford, 1920.

8. Hollcroft, T. R., *Singularities of curves of given order*, this Bulletin, vol. 29 (1923), pp. 407–414.

9. Hollcroft, T. R., *Maximal cuspidal curves*, Annals of Mathematics, (2), vol. 26 (1924), pp. 37–46.

10. Hollcroft, T. R., *On the reality of singularities of plane curves*, Mathematische Annalen, vol. 97 (1927), pp. 775–787.

11. Hollcroft, T. R., *Invariants associated with singularities of algebraic curves*, Acta Mathematica, vol. 56 (1930), pp. 261–272.

12. Klein, F., *Eine neue Relation zwischen den Singularitäten einer algebraischen Kurve*, Mathematische Annalen, vol. 10 (1876), pp. 199–209.

13. Lefschetz, S., *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–41.

14. del Pezzo, P., *Equazione d'una curva del quinto ordine dotata di cinque cuspidi*, Rendiconti, Accademia delle Scienze Fisiche di Napoli, (2), vol. 3 (1889), pp. 46–49.

15. Plücker, J., *Theorie der algebraischen Curven*, Bonn, 1839.

16. Segre, B., *Esistenza e dimensione di sistemi continui di curve piane algebriche con dati caratteri*, Lincei Rendiconti, (6), vol. 10 (1929), pp. 31–38.

17. Segre, B., *Esistenza di sistemi continui distinti di curve piane algebriche con dati numeri plueckeriani*, Lincei Rendiconti, (6), vol. 10 (1929), pp. 557–560.

18. Segre, B., *Sulla caratterizzazione delle curve di diramazione de piani multipli generali*, Memorie, Reale Academia d'Italia, Classe di Scienze Fisiche, Matematiche e Naturali, vol. 1 (1930), pp. 5–31.

19. Severi, F., *Vorlesungen über algebraische Geometrie*, Severi-Löffler, Berlin, 1921, Anhang F, pp. 307–353.

20. Snyder, V., *Construction of plane curves of given order and genus, having distinct double points*, this Bulletin, vol. 15 (1908), pp. 1–4.

21. Zariski, O., *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, American Journal of Mathematics, vol. 51 (1929), pp. 305–328.

22. Zariski, O., *On the linear connection index of the algebraic surfaces* $z^n=f(x, y)$, Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 494–501.

23. Zariski, O., *On the non-existence of curves of order* 8 *with* 16 *cusps*, American Journal of Mathematics, vol. 53 (1931), pp. 309–318.

24. Zariski, O., *On the irregularity of cyclic multiple planes*, Annals of Mathematics, (2), vol. 32 (1931), pp. 485–511.

25. Zariski, O., *Algebraic Surfaces*, Ergebnisse der Mathematik, vol. 3 (1935), pp. 160–181.

WELLS COLLEGE

JENSEN'S INEQUALITY*

BY E. J. MCSHANE

The simplest form of Jensen's inequality is that if $\phi(x)$ is a convex function, and m is the arithmetic mean of $x_1, \cdots, x_n$, then the mean of the numbers $\phi(x_n)$ is not less than $\phi(m)$. This inequality can be generalized in several different ways. The function $\phi(x)$ can be replaced by a convex function of several variables, and the arithmetic mean can be replaced by any one of several other means, as has been shown in various proofs. Since the inequality is of considerable utility, it seems worth while to have it established in a form which is general enough to cover a wide assortment of applications.

The proofs will rest on two well known properties of convex sets.† If K is closed and convex and a point p is not in K, then p can be separated from K by a hyperplane. If K is closed and convex and p is a boundary point of K, there is a hyperplane of support of K passing through p.

1. *The Inequality in Geometric and in Analytic Form.* It will be convenient in the following proofs to use these symbols and definitions:

R_n is n-dimensional Euclidean space. Its points will be denoted by $(z_1, \cdots, z_n)$ or by z. Linear functions $\sum a_i z_i$ or R_n will be symbolized by $l(z)$.

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* Presented to the Society, December 31, 1936.

† A set is convex if for every pair P, Q of points of the set the line segment PQ is contained in the set.

L is a linear class of real valued functions $f(x)$ defined on a set E. It shall be supposed have the properties:

(1a) If f_1, f_2 are in L and k_1, k_2 are real numbers, $k_1f_1+k_2f_2$ is in L.

(1b) The function defined and constantly equal to 1 on E is in L.

Mf is a linear mean defined on L, with these properties:

(2a) $M1=1$.

(2b) If f_1, f_2 are in L and k_1, k_2 are real numbers, $M(k_1f_1+k_2f_2) = k_1Mf_1+k_2Mf_2$.

(2c) If $f(x)$ is in L and $f(x)\geqq 0$, then $Mf\geqq 0$.

If $\boldsymbol{f}(x)$ is an n-tuple of functions $(f_1(x), \cdots, f_n(x))$ of L, we denote by $\boldsymbol{Mf}$ the n-tuple $(Mf_1, \cdots, Mf_n)$. From (2) we obtain

(3) $Ml(\boldsymbol{f}) = l(\boldsymbol{Mf})$ for every function $l(\boldsymbol{z})$ linear on R_n.

The geometric formulation of Jensen's inequality is as follows:

THEOREM 1. *Let* (1) *and* (2) *be satisfied. Let K be a closed convex point set in R_n. Let $f_1(x), \cdots, f_n(x)$ be functions of the class L such that $\boldsymbol{f}=(f_1, \cdots, f_n)$ is in K for all x in E. Then $\boldsymbol{Mf}$ is in K.*

Let $l(\boldsymbol{z})+c=0$ be a hyperplane in R_n such that K is entirely to one side, say $l(\boldsymbol{z})+c\geqq 0$ for $\boldsymbol{z}$ in K. Then $l(\boldsymbol{f})+c\geqq 0$ for all x, and $0\leqq M(l(\boldsymbol{f})+c) = Ml(\boldsymbol{f})+Mc = l(\boldsymbol{Mf})+c$, so that $\boldsymbol{Mf}$ lies on the same side of the hyperplane as K. That is, no hyperplane separates $\boldsymbol{Mf}$ from K. This is only possible if $\boldsymbol{Mf}$ is in K.

The more usual analytical formulation of the inequality is covered by the following theorem:

THEOREM 2. *Let* (1) *and* (2) *be satisfied. Let K be a closed convex point set in R_n, and let $\phi(\boldsymbol{z})$ be continuous and convex* on K. Let $f_1(x), \cdots, f_n(x)$ be functions of the class L such that $\boldsymbol{f}(x) = (f_1(x), \cdots, f_n(x))$ is in K for all x in E, and such, moreover, that $\phi(\boldsymbol{f}(x))$ is in the class L. Then $\phi(\boldsymbol{Mf})$ is defined and*

$$(J) \qquad \phi(\boldsymbol{Mf}) \leqq M\phi(\boldsymbol{f}).$$

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* We call ϕ convex on K if $\phi(\frac{1}{2}(\boldsymbol{z}_1+\boldsymbol{z}_2)) \leqq \frac{1}{2}(\phi(\boldsymbol{z}_1)+\phi(\boldsymbol{z}_2))$ for all $\boldsymbol{z}_1, \boldsymbol{z}_2$ in K.

Denote the points $(z_1, \cdots, z_{n+1})$ of R_{n+1} by z^1. These can also be denoted by (z, z_{n+1}), where z is in R_n. Define K_1 to be the set in R_{n+1} of points (z, z_{n+1}) such that z is in K and $z_{n+1} \geqq \phi(z)$. It is clear that K_1 is closed and convex, and for all x in E the point $(f(x), \phi(f(x)))$ is in K_1. Hence by Theorem 1, the point $(Mf, M\phi(f))$ is in K_1; that is, Mf is in K and $M\phi(f) \geqq \phi(Mf)$.

Obviously we could weaken our hypotheses somewhat by omitting the requirement that ϕ be continuous and assuming instead that the set K_1 is closed and convex. This would permit $\phi(z)$ to have infinite discontinuities on the boundary of K.

EXAMPLES: (1) Let $\alpha(x_1, \cdots, x_m)$ be a positively monotonic* function of the variables $(x_1, \cdots, x_m)$, and let E be a set measurable with respect to α and such that $0 < m_\alpha E < \infty$. Let L be the class of all functions $f(x)$ which are Lebesgue-Stieltjes integrable with respect to α over E. Let $Mf = \int_E f d\alpha / \int_E d\alpha$. Then conditions (1) and (2) are satisfied. The conclusion of Theorem 2 assumes the form

$$\phi\left(\int f_1 d\alpha \Big/ \int d\alpha, \cdots, \int f_n d\alpha \Big/ \int d\alpha\right) \leqq \int \phi(f_1, \cdots, f_n) d\alpha \Big/ \int d\alpha.$$

In the next three examples requirements of convergence or integrability are too obvious to need statement:

(2) The range of x is $(1, \cdots, m)$ or $(1, 2, \cdots)$, so that $f(x)$ is a (finite or infinite) sequence $(a_1, a_2, \cdots)$, and $Mf = \sum c_i a_i / \sum c_i$, where $c_i \geqq 0$ and $0 < \sum c_i < \infty$.

(3) The functions f of L are continuous, and $Mf = \int_E dx / \int_E dx$.

(4) The functions f of L are Lebesgue measurable over the measurable set E, and $Mf = \int f p dx / \int p dx$, where $p(x) \geqq 0$ and $0 < \int p dx < \infty$.

(5) E is in the interval $(0,1)$, L is the class of *all* bounded functions on E, Mf is the Banach integral† of f over $(0, 1)$.

(6) E is the set of all real numbers, L the class of all uniformly almost periodic functions, Mf is the mean value of f.

* That is, a function whose mth difference is non-negative.

† Banach, *Théorie des Opérations Linéaires*, p. 31.

(7) More generally, E is any group, L is the class of all functions almost periodic on E, Mf is von Neumann's* mean value of f.

(8) In the space $(x_1, x_2, \cdots)$ of infinitely many dimensions, $(0 \leqq x_i \leqq 1)$, E is a set of finite positive measure,† L is the class of functions summable over E, $Mf = \int_E f dx/mE$.

2. *Conditions for Strict Inequality.* In §1 we have not mentioned conditions for strict inequality. To investigate this question it is convenient to define negligible sets. A set $S \subset E$ is negligible (with respect to L and M) if there exists a function f in the class L such that

(a) $f(x) \geqq 0$ on E, (b) $f(x) > 0$ on S, (c) $Mf = 0$.

It follows readily that every subset of a negligible set is negligible, and so is every set which is the sum of a finite number of negligible sets. It is then easy to prove the following theorem:

THEOREM 3. *In Theorem* 1, $\boldsymbol{Mf}$ *is a boundary point of* K *only if all points* $\boldsymbol{f}(x)$ *except those corresponding to a negligible set of* x *belong to the intersection of* K *with one of its hyperplanes of support.*

If $\boldsymbol{Mf}$ is a boundary point of K, through it there passes a hyperplane of support π: $l(z)+c=0$ of K; say $l(z)+c \geqq 0$ for z in K. Let S be the set of x such that $\boldsymbol{f}(x)$ is not in the intersection πK; then $l(\boldsymbol{f}(x))+c>0$ on S. Since $l(\boldsymbol{f}(x))+c \geqq 0$ for all x and $M(l(\boldsymbol{f}(x))+c) = l(\boldsymbol{Mf})+c=0$, we see that S is negligible.

I omit the easy proof of the following theorem:

THEOREM 4. *If the set* K *is strictly convex, and* ϕ *is strictly convex on* K, *then in Theorem* 2 *equality holds only if* $f_i(x) = Mf_i = \text{const.}$ $(i=1, \cdots, n)$ *except on a negligible set.*

I am unable to state whether the conditions $f_i = Mf_i$ except on a negligible set are sufficient as well as necessary for equality in Theorem 4. However, by adding a further postulate concern-

* J. von Neumann, *Almost periodic functions in a group*, Transactions of this Society, vol. 36 (1934); in particular, p. 452.

† P. J. Daniell, *Integrals in an infinite number of dimensions*, Annals of Mathematics, (2), vol. 20 (1919), p. 281.

B. Jessen, *The theory of integration in a space of an infinite number of dimensions*, Acta Mathematica, vol. 63 (1934), p. 249.

ing L and M we can establish this, even when K is not strictly convex. We henceforth restrict our attention to systems L, M such that the following condition holds:

(4) If S is any negligible subset of E, and $f(x)$ is any (real) function which vanishes on $E-S$, then $f(x)$ is in the class L and $Mf=0$.

In example (1) negligible sets are sets of measure $m_a S=0$; hence condition (4) is satisfied. In example (2), a set S of integers is negligible if $\sum_{i\in S} c_i=0$; in (4), S is negligible if $p(x)=0$ on almost all of S. In (8), S is negligible if $mS=0$. In (3), (6), and (7), only the empty set is negligible. For all of these (4) is valid. I do not know whether example (5) satisfies (4).

An immediate consequence of conditions (1), (2), and (4) is that if $f(x)$ is any function of class L, and $g(x)=f(x)$ except on a negligible set, then $g(x)$ is in L and $Mf=Mg$.

THEOREM 5. *If condition* (4) *is satisfied, then in inequality* (*J*) *equality holds if and only if the following condition holds: For all x except at most those belonging to a negligible set S, the point $(f_1(x), \cdots, f_n(x))$ belongs to a convex subset K' of K on which $\phi(z)$ is linear. In particular, if $\phi(z)$ is strictly convex* equality holds if and only if $f_i(x)=$ const. except on a negligible set.*

The last statement is an immediate consequence of the first, for if ϕ is strictly convex the only subsets K' on which ϕ is linear consist of single points. Suppose then that the condition of Theorem 5 holds; by redefining $f_i(x)$ on S, it will be true that $\boldsymbol{f}$ is in K' for all x, without change in $\boldsymbol{Mf}$ or $M(\phi(\boldsymbol{f}))$. On K' we have $\phi(z)=l(z)+c$; hence $M\phi(\boldsymbol{f})=M(l(\boldsymbol{f})+c)=l(\boldsymbol{Mf})+c$. But since K' is convex, by Theorem 1 the point $\boldsymbol{Mf}$ is in K', and so $\phi(\boldsymbol{Mf})=l(\boldsymbol{Mf})+c$. Hence equality holds in the inequality (*J*).

To prove the necessity of our condition we first observe that there may be linear relations $l(\boldsymbol{f}(x))+c=0$ holding for all x except those of a negligible set. We choose a maximal set of such linear relations; there is no loss of generality in assuming that these are of the form $f_{s+1}(x)=\cdots=f_n(x)=0$ except on S_1, where S_1 is negligible. Then $Mf_{s+1}=\cdots=Mf_n=0$.

We now change notation. Let R_s be the space of points $(z_1, \cdots, z_s)$; let H be the set of $(z_1, \cdots, z_s)$ such that $(z_1, \cdots, z_s, 0, \cdots, 0)$ is in K; let $\psi(z_1, \cdots, z_s)=\phi(z_1, \cdots, z_s,$

* We assume only that K is convex, not that it is strictly convex.

$0, \cdots, 0)$ on H; in the space R_{s+1} of points $(z_1, \cdots, z_s, z_{n+1})$ let H_1 be the set for which $(z_1, \cdots, z_s)$ is in H and $z_{n+1} \geqq \psi(z_1, \cdots, z_s)$. For x in $E - S_1$ we have $(f_1(x), \cdots, f_s(x))$ in H and $(f_1, \cdots, f_s, \psi(f_1, \cdots, f_s))$ in H_1. If equality holds in (J), then

$$\begin{aligned} M\psi(f_1, \cdots, f_s) &= M\phi(f_1, \cdots, f_s, 0, \cdots, 0) = M\phi(f_1, \cdots, f_n) \\ &= \phi(Mf_1, \cdots, Mf_s, 0, \cdots, 0) \\ &= \psi(Mf_1, \cdots, Mf_s), \end{aligned}$$

so the point $(Mf_1, \cdots, Mf_s, M\psi(f_1, \cdots, f_s))$ is a boundary point of H_1 in R_{s+1}.

By Theorem 3, for all points x of $E - S_1$ except a negligible set S_2 the point $(f_1(x), \cdots, f_s(x), \psi(f_1, \cdots, f_s))$ belongs to the intersection of H_1 with a hyperplane of support π: $a + b_1 z_1 + \cdots + b_s z_s + c z_{n+1} = 0$. That is, except on $S_1 + S_2$ the equation $a + b_1 f_1 + \cdots + b_s f_s + c\psi(f_1, \cdots, f_s) = 0$ holds. Here $c \neq 0$; otherwise we would have a new linear relation between the f_i independent of the maximal set $f_{s+1} = \cdots = f_n = 0$. We may therefore suppose that $c = 1$; hence π has the equation $a + \sum b_i z_i + z_{n+1} = 0$. The left member of this equation is positive for some points $(z_1, \cdots, z_s, z_{n+1})$ of H_1, since z_{n+1} is arbitrarily large. But π is a hyperplane of support, so $a + \sum b_i z_i + z_{n+1}$ does not change sign on H_1; therefore $a + \sum b_i z_i + z_{n+1} \geqq 0$ on H_1. For $(z_1, \cdots, z_s)$ in H the point $(z_1, \cdots, z_s, \psi(z_1, \cdots, z_s))$ is in H_1, so that $a + \sum b_i z_i + \psi(z_1, \cdots, z_s) \geqq 0$ on H. Moreover, if $z_{n+1} > \psi(z_1, \cdots, z_s)$, then $a + \sum b_i z_i + z_{n+1} > 0$. Hence the points of H_1 which lie in π are those of the form $(z_1, \cdots, z_s, \psi(z_1, \cdots, z_s))$ with $(z_1, \cdots, z_s)$ in H and $a + \sum b_i z_i + \psi(z_1, \cdots, z_s) = 0$. The points satisfying these conditions form the intersection πH_1, which, being the intersection of convex sets, must be convex, as is also its projection H' on the space R_s. For all x in $E - (S_1 + S_2)$ the point $(f_1(x), \cdots, f_s(x), \psi(f_1, \cdots, f_s))$ is in πH_1, so that $(f_1(x), \cdots, f_s(x))$ is in H'. If we define K' to be the set of all points $(z_1, \cdots, z_s, 0, \cdots, 0)$ with $(z_1, \cdots, z_s)$ in H', then K' is convex, and for all x in $E - (S_1 + S_2)$ the point $(f_1(x), \cdots, f_s(x), 0, \cdots, 0)$ is in K'. If $(z_1, \cdots, z_n)$ is in K' then $(z_1, \cdots, z_s)$ is in H', and

$$\begin{aligned} \phi(z_1, \cdots, z_n) &= \phi(z_1, \cdots, z_s, 0, \cdots, 0) \\ &= \psi(z_1, \cdots, z_s) = -a - \sum b_i z_i. \end{aligned}$$

This establishes the theorem.

3. *Extension to Banach Spaces.* It is possible to extend Theorems 1, 2, 3, and 4 to functions $f(x)$ assuming values in a Banach space B. Suppose that (1) and (2) are satisfied, and that L is a class of functions $f(x)$ defined on E and assuming values in B. We shall assume that for every linear function $l(z)$ on B and every $f(x)$ in L the function $l(f(x))$ is in L. Further, we shall assume that there is a linear mean M defined on L such that for every linear function $l(z)$ on B and every f in L we have

$$l(Mf) = M(l(f)).$$

We then find that Theorems 1, 2, 3, and 4 extend with only one change. The only properties of convex sets which we used were these: through each boundary point of a convex set there passes a hyperplane of support, and each point which does not belong to a convex set can be separated from it by a hyperplane. These properties have been established for convex *bodies** (closed convex sets having interior points). Hence our theorems extend at once, provided that we replace the words "convex set" by "convex body."

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* Cf. Banach, *Théorie des Opérations Linéaires*, p. 246.

BILINEAR FUNCTIONALS ON THE SPACE OF BOUNDED, MEASURABLE FUNCTIONS

BY H. H. GOLDSTINE

1. *Introduction.* We propose to make an extension of the Radon integral to two variables, and by properly defining the notion of limited variation to show that every continuous bilinear functional on the class of bounded, measurable functions (with suitably defined norm) is expressible in terms of this integral. Conversely we show that the integral is bilinear and continuous. The paper is concluded with a theorem establishing a relation between the double and iterated integrals. Similar results for the case of continuous linear functionals have recently been obtained.*

2. *Preliminary Definitions.* Let $\mathfrak{S}$ be the class of all measurable subsets of the interval $\mathfrak{I} = [0 \leqq x \leqq 1]$, and let ϕ be a function on $\mathfrak{S}\mathfrak{S}$ to $\mathfrak{A}$, the set of all real numbers. Such a function will be termed *bi-additive* in case, for every two disjoint sets σ_1, σ_2, we have $\phi(\sigma, \sigma_1+\sigma_2) = \phi(\sigma, \sigma_1)+\phi(\sigma, \sigma_2)$ and $\phi(\sigma_1+\sigma_2, \sigma) = \phi(\sigma_1, \sigma)+\phi(\sigma_2, \sigma)$; the function ϕ will be said to be of *limited variation* in case the upper bound of

$$\left| \sum_{i,j} \epsilon_{1i}\phi(\sigma_{1i}, \sigma_{2j})\epsilon_{2j} \right| \tag{1}$$

for all divisions $\sigma_{k_1}, \cdots, \sigma_{km_k}$, $(k=1, 2)$, of $\mathfrak{I}$ into disjoint sets and all numbers ϵ_{ki} such that $|\epsilon_{ki}| = 1$, $(k=1, 2; i=1, 2, \cdots, m_k)$, is finite;† we shall designate this bound, when it is finite, by $T\phi$.

Let μ_1, μ_2 be any two functions in $\mathfrak{M}$, the class of bounded, measurable functions, such that $d_{k0} \leqq \mu_k(x) < D_k$, $(k=1, 2; 0 \leqq x \leqq 1)$. Then following the usual procedure in the Lebesgue

* See, T. H. Hildebrandt, *On bounded linear functional operations*, Transactions of this Society, vol. 36 (1934), p. 868; or G. Fichtenholz and L. Kantorovitch, *Sur les opérations dans l'espace des fonctions bornées*, Studia Mathematica, vol. 5 (1934), p. 69.

† This definition is a generalization to set functions of Fréchet's definition for functions of real variables; see, M. Fréchet, *Sur les fonctionelles bilinéaires*, Transactions of this Society, vol. 16 (1915), p. 215.

theory, we divide (d_{k0}, D_k) into sub-intervals by numbers $d_{k0} < d_{k1} < \cdots < d_{km_k} = D_k$, $(k = 1, 2)$; this process gives rise to a *partition* π of the rectangle determined by d_{k0} and D_k. Throughout this paper we shall use the term partition in this restricted sense. To simplify further the notation let σ_{ki} be the set of all x such that $d_{ki} \leqq \mu_k(x) < d_{ki+1}$. Finally if the limit, as the norm of π approaches zero, of

$$s(\pi) = \sum_{\pi} \xi_{1i}\xi_{2j}\phi(\sigma_{1i}, \sigma_{2j}) \tag{2}$$

exists, where $d_{ki} \leqq \xi_{ki} \leqq d_{ki+1}$, it will be designated by

$$\int_0^1 \int_0^1 \mu_1(s)\mu_2(t)\phi(d\mathfrak{F}, d\mathfrak{F}). \tag{3}$$

3. *The Existence of the Integral.* We proceed to a statement of a sufficient condition for the existence of the integral.

THEOREM 1. *If the set function ϕ on $\mathfrak{S}\mathfrak{S}$ to $\mathfrak{A}$ is of limited variation and bi-additive, the double integral* (3) *exists for every pair μ_1, μ_2 of functions in $\mathfrak{M}$.*

The proof of this theorem follows the usual lines of demonstration for the existence of a Lebesgue integral of a bounded function.

We shall associate with the class $\mathfrak{M}$ two norms $\| \ \|_1$ and $\| \ \|_2$ which will thus give rise to two normed linear spaces $\mathfrak{M}_k = (\mathfrak{M}, \| \ \|_k)$, $(k = 1, 2)$. These norms are defined as follows:*

$$\|\mu\|_1 \equiv \overline{B}_t |\mu(t)|,$$

and

$$\|\mu\|_2 \equiv \operatorname*{true\,max}_{0 \leq t \leq 1} |\mu(t)| = \lim_{n=\infty} \left[\int_0^1 |\mu(t)|^n dt\right]^{1/n}.$$

4. *The Representation Theorem.* It should be noted that if a functional K is bilinear and continuous on $\mathfrak{M}_2\mathfrak{M}_2$, it also has these properties as on $\mathfrak{M}_1\mathfrak{M}_1$.

THEOREM 2. *If K on $\mathfrak{M}_1\mathfrak{M}_1$ to $\mathfrak{A}$ is bilinear and continuous, then there exists a function ϕ on $\mathfrak{S}\mathfrak{S}$ to $\mathfrak{A}$ which is bi-additive and of limited variation such that*

* See Fichtenholz and Kantorovitch, loc. cit., p. 69.

$$K(\mu_1, \mu_2) = \int_0^1 \int_0^1 \mu_1(s)\mu_2(t)\phi(d\mathfrak{I}, d\mathfrak{I}), \tag{4}$$

for every μ_1, μ_2 in $\mathfrak{M}_1$; moreover the modulus of K is the total variation $T\phi$ of ϕ. If K is bilinear and continuous on $\mathfrak{M}_2\mathfrak{M}_2$, then ϕ has the additional property that the measure of σ_1 or $\sigma_2 = 0$ implies that $\phi(\sigma_1, \sigma_2) = 0$.

Indicating, as before, the characteristic function of a set σ by μ_σ, we define $\phi(\sigma_1, \sigma_2)$ to be $K(\mu_{\sigma_1}, \mu_{\sigma_2})$. The first property follows immediately from the bilinearity of K; the second results from the continuity of K as follows:

$$\begin{aligned} \left| \sum_{i,j} \epsilon_{1i}\phi(\sigma_{1i}, \sigma_{2j})\epsilon_{2j} \right| &= \left| K\left(\sum_i \epsilon_{1i}\mu_{\sigma_{1i}}, \sum_j \epsilon_{2j}\mu_{\sigma_{2j}} \right) \right| \\ &\leqq M(K) \left\| \sum_i \epsilon_{1i}\mu_{\sigma_{1i}} \right\|_1 \cdot \left\| \sum_j \epsilon_{2j}\mu_{\sigma_{2j}} \right\|_1 \leqq M(K); \end{aligned} \tag{5}$$

furthermore, if K is continuous on $\mathfrak{M}_2\mathfrak{M}_2$, then $|\phi(\sigma_1, \sigma_2)| = |K(\mu_{\sigma_1}, \mu_{\sigma_2})| \leqq M(K)\|\mu_{\sigma_1}\|_2 \cdot \|\mu_{\sigma_2}\|_2 = 0$ if $m\sigma_1 = 0$ or $m\sigma_2 = 0$. It follows from inequality (5) that $T\phi \leqq M(K)$ and by the usual argument that the equality holds.

Let μ_1, μ_2 be any two bounded, measurable functions. Denoting by μ_{1n} and μ_{2n} the sums

$$\sum_{i=0}^{n-1} d_{1i}\mu_{\sigma_{1i}} \qquad \text{and} \qquad \sum_{j=0}^{n-1} d_{2j}\mu_{\sigma_{2j}},$$

where d_{ki} and σ_{ki} have the same significance as before, we see that

$$K(\mu_{1n}, \mu_{2n}) = \sum_{i,j} d_{1i}d_{2j}\phi(\sigma_{1i}, \sigma_{2j});$$

the result then follows by the continuity of K.

It is of some interest to note that when ϕ has the properties enumerated above, the functional defined by equation (4) is linear and continuous.

5. *The Iterated Integrals.* To establish the existence of the iterated integrals it will be shown that the Radon integral

$$\Phi(\sigma) = \int_0^1 \mu_2(t)\phi(\sigma, d\mathfrak{I}) \tag{6}$$

exists, is additive, and is of limited variation. To do this let σ_c be the complement of σ and $\sigma_1, \cdots, \sigma_n$ be any division of $\mathfrak{I}$ into disjoint sets. Then, if ϕ is of limited variation, we have

$$\left| e\sum_{j=1}^{n} \phi(\sigma, \sigma_j)\epsilon_j + e_c\sum_{j=1}^{n} \phi(\sigma_c, \sigma_j)\epsilon_j \right| \leqq T\phi;$$

hence taking ϵ_j to be the sign of $\phi(\sigma, \sigma_j)$, e to be 1, and e_c the sign of $\sum\phi(\sigma_c, \sigma_j)\epsilon_j$, we have

$$\sum_{j=1}^{n} \left| \phi(\sigma, \sigma_j) \right| \leqq T\phi.$$

This establishes the existence of $\Phi(\sigma)$, for every σ. The additivity of Φ is obvious. Hence we have the following theorem:

THEOREM 3. *If ϕ on $\mathfrak{S}\mathfrak{S}$ to $\mathfrak{A}$ is bi-additive and of limited variation, then the iterated integrals*

$$(7) \qquad \int_0^1 \mu_1(s) \int_0^1 \mu_2(t)\phi(d\mathfrak{I}, d\mathfrak{I}) \quad \text{and} \quad \int_0^1 \mu_2(t) \int_0^1 \mu_1(s)\phi(d\mathfrak{I}, d\mathfrak{I})$$

exist and equal the double integral.

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ON LOCALLY COMPACT METRISABLE SPACES

BY H. E. VAUGHAN*

The object of this paper is to present two characterizations, one of the class of locally compact metrisable spaces, the other of the class of locally compact, separable, metrisable spaces (locally compact, perfectly separable Hausdorff spaces), as follows.

THEOREM 1. *In order that a metrisable space be locally compact it is necessary and sufficient that it be the difference of two closed sets in every metric space in which it is topologically imbedded.*†

THEOREM 2. *In order that a Hausdorff space be homeomorphic to a totally complete metric space it is necessary and sufficient that it be locally compact and perfectly separable.*‡

In Theorem 1 the word metric may be replaced, on the one hand, by regular, on the other, by complete metric. This theorem is of interest chiefly because of the similar well known characterizations of the class of all compact metrisable spaces and of the class of all metrisable spaces which are homeomorphic to complete metric spaces.§ The method of proof also relates it to the two characterizations of the class of compact metrisable

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† A distinction is made between the adjectives metric and metrisable. The word metric implies the definite choice of a metric in a metrisable space. The phrase *A is topologically imbedded in B* means that *A* is homeomorphic to a subset of *B*.

‡ A totally complete metric space is a metrisable space in which the metric is so chosen that every bounded set is compact. See K. Menger, *Über die Dimension von Punktmengen*, Monatshefte für Mathematik und Physik, vol. 34 (1926), pp. 135–161; in particular, p. 144.

§ The first of these theorems is obtained by replacing, in Theorem 1, "locally compact" by "compact" and "the difference of two closed sets" by "closed." The second is obtained by replacing "locally compact" by "homeomorphic to a complete metric space" and "the difference of two closed sets" by "the intersection of a sequence of open sets." A proof of the first may be given along the lines of the proof of the first lemma, stated below. For a proof of the second, see C. Kuratowski, *Topologie I*, p. 215.

spaces, as those which are bounded (complete) in every metric.* Theorem 2 is also related to the second theorem mentioned in the footnote above. Its chief interest, however, seems to lie in the fact that the class of spaces characterized was used to a considerable extent by K. Menger† and W. Sierpinski‡ but apparently has not before been characterized in non-metric terms.

Theorem 1 is an almost immediate consequence of the following two lemmas.

LEMMA 1. *A metrisable space which is locally closed at each of its points§ in every complete metric space in which it is topologically imbedded is locally compact.*

PROOF. Let M be a metrisable space which is not locally compact. Supposing that a metric has been chosen, there exists a point a of M such that‖ $\overline{S(a, 1/n)}$ is not compact for any positive integer n. Consequently, there exists a denumerable family $((a_{ni}))$ of sequences of points of M such that, for every i, a_{ni} belongs to $S(a, 1/n)$ and the set $a+\sum_{n,i}a_{ni}$ is a closed subset of M. Hence,¶ a metric may be chosen for M such that the set $a+\sum_{n,i}a_{ni}$ is isometric to the linear set $(0)+\sum_{n,i}(1/2^n-1/2^{n+i})$, where (x) denotes the point whose abscissa is x. Now** M may be imbedded isometrically in a complete metric space $\tilde{M}$ and, since each of the sequences (a_{ni}) now satisfies the Cauchy convergence criterion and all but a finite number of the sequences are contained in any neighborhood of the point a, M is not locally closed at the point a in $\tilde{M}$.

LEMMA 2. *A locally compact metrisable space is locally closed at each of its points in every Hausdorff space in which it is topologically imbedded.*

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* See F. Hausdorff, *Erweiterung einer Homöomorphie*, Fundamenta Mathematicae, vol. 16 (1930), pp. 352–360; in particular, p. 359.

† Loc. cit.

‡ See W. Sierpinski, *General Topology*, p. 118 et seq.

§ A subset M of a neighborhood space R is said to be locally closed at the point p if there exists a neighborhood of p whose intersection with M is a closed set. See Kuratowski, loc. cit., p. 65.

‖ If a is a point of a metric space M and d is a positive real number, $S(a, d)$ denotes the set of all points of M whose distance from a is less than d and $\overline{S(a, d)}$ denotes this set plus its limit points.

¶ See F. Hausdorff, loc. cit.

** See Kuratowski, loc. cit., p. 200.

PROOF. Each point of a locally compact metrisable space has a neighborhood which is a compact metrisable space. Such a neighborhood is closed in every Hausdorff space in which it is topologically imbedded.

PROOF OF THEOREM 1. This results from the preceding lemmas and the fact that a subset of a regular space which is locally closed at each of its points is the difference of two closed sets, and conversely.*

PROOF OF THEOREM 2. NECESSITY. Let M be a totally complete metric space. Since every point of M has a bounded neighborhood, M is certainly locally compact, while, since $M=\sum_1^\infty M_n$, where M_n is the bounded set of points of M whose distance from some fixed point a_0 of M is at most n, M is the sum of a denumerable family of compact metric spaces and hence is perfectly separable.

SUFFICIENCY. Let M be a locally compact, perfectly separable, Hausdorff space. Then† there exists a perfectly separable compact, Hausdorff space $\tilde{M}$ and a point a_0 of $\tilde{M}$ such that $M=\tilde{M}-(a_0)$. Since $\tilde{M}$ is a separable metric space it may be considered as a subset of the Hilbert parallelotope‡ and its diameter may be assumed to be at most equal to 1. Let the coordinates of an arbitrary point x of $\tilde{M}$ be denoted by (x_i). Then it may be assumed that $x_1=0$, $(0\leqq x_i\leqq 1/i, i\neq 1)$. The sequence $(f_i'(x))$, where $f_1'(x)=1-d(x, a_0)$ and $f_i'(x)=x_i$, $(i\neq 1)$, is evidently a (1-1) continuous mapping of $\tilde{M}$ on a subset of the Hilbert parallelotope and, since $\tilde{M}$ is compact, is a homeomorphism. The sequence $(f_i(x))$, where

$$f_1(x)=\frac{f_1'(x)}{1-f_1'(x)}, \qquad f_i(x)=f_i'(x), \qquad (i\neq 1),$$

is clearly also a homeomorphism of $\tilde{M}-(a_0)$ on a subset of Hil-

* See Kuratowski, loc. cit., p. 65.

† See P. Alexandroff, *Über die Metrisation der im kleinen kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294–301.

‡ In this connection, by Hilbert space is meant the metric space whose elements are the infinite sequences, (x_i), of real numbers such that $\sum_1^\infty x_i^2<\infty$. The distance, $d(x, y)$, between the points (x_i) and (y_i) is given by the formula $d(x, y)=(\sum_1^\infty (x_i-y_i)^2)^{1/2}$. The Hilbert parallelotope is the set of points of Hilbert space such that, for every i, $0\leqq x_i\leqq 1/i$.

bert space. The function $d'(x, y) = d(f(x), f(y))$, where x and y are arbitrary points of M, may evidently be taken as a metric for M and satisfies the theorem, for a subset of M which is non-compact contains points x such that $f_1(x)$ is arbitrarily large.

COROLLARY. *Any locally compact metrisable space is homeomorphic to a metric space in which every bounded, connected, set is compact.*

PROOF. Let M be a locally compact metrisable space. Then M is a sum of disjoint open sets each of which is a locally compact, separable, metrisable space.* By Theorem 2, each of these is homeomorphic to a totally complete metric space. The latter may be combined in the following manner to form a metric space of the sort described in the corollary. In each of the above mentioned spaces choose some definite point and call it the *origin* of the space. Let M_1, M_2, and M_3 denote any three of the above spaces, let d_1, d_2, and d_3 denote the respective metrics, and let o_1, o_2, and o_3 denote the respective origins. If x and y are points of the same space, say M_1, define $d(x, y)$ as $d_1(x, y)$. If x and y belong to different spaces, say M_1 and M_2, respectively, define $d(x, y)$ as $d_1(x, o_1)+d_2(y, o_2)+1$. The function $d(x, y)$ so defined is easily seen to be a metric which defines a space homeomorphic to M in which every bounded connected set is compact.

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* See P. Alexandroff, loc. cit.

ON SOME GAP THEOREMS FOR EULER'S METHOD OF SUMMATION OF SERIES

BY YOSHITOMO OKADA

Hardy and Littlewood* have proved the following theorem:

For a given series $\sum_{k=1}^{\infty} a_{n_k}$, $(a_{n_k} \neq 0)$, *let* θ *be a fixed constant such that*

$$\frac{n_{k+1}}{n_k} \geqq \theta > 1, \qquad (k = 1, 2, \cdots).$$

If this series be summable by Abel's method of summation to the sum s, *then this series is convergent and its sum is* s.

Obreschkoff† obtained also a similar result for Cesàro's method. We shall now study these results for Euler's method.

We shall begin with the following theorem:

THEOREM 1.‡ *Let* $\sum_{n=0}^{\infty} a_n$ *be a given series summable by Euler's method, that is, if* $s_0 = 0$, $s_n = a_0 + a_1 + \cdots + a_{n-1}$, $(n \geqq 1)$,

$$\lim_{n\to\infty} \frac{1}{2^n}\left\{s_0 + n s_1 + \frac{n(n-1)}{2!} s_2 + \cdots + s_n\right\} = s \tag{1}$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n_k'\}$, $(n_k < n_k')$, *of integers and for a given number* α, $(1 \leqq \alpha < 2)$, *let*

$$\begin{aligned} &a_\nu = 0, \text{ for } n_k < \nu < n_k', \quad (k = 1, 2, \cdots), \\ &a_n = O(\alpha^n). \end{aligned} \tag{2}$$

If $\eta_k'/\eta_k \geqq (1+\eta)/(1-\eta)$, $(k = 1, 2, \cdots)$, *for a positive number* η *such that*

$$(1+\eta)\log(1+\eta) + (1-\eta)\log(1-\eta) - 2\log\alpha > 0,$$

then

$$\lim_{k\to\infty} \sum_{\nu=0}^{n_k} a_\nu = s. \tag{3}$$

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* Hardy and Littlewood, Proceedings of the London Mathematical Society, (2), vol. 25 (1926).

† Obreschkoff, Tôhoku Mathematical Journal, vol. 32 (1930).

‡ If, in this theorem, (1) holds uniformly and O of (2) is independent of z when each a_n is a function of z, then (3) also holds uniformly.

PROOF. To prove this, we can consider that all $n_k' - n_k - 1$ are even. Then putting

$$n_k + \frac{n_k' - n_k - 1}{2} + 1 = m$$

we have

$$a_{m-1} = a_{m-2} = \cdots = a_{n_k+1} = 0,$$

$$a_m = a_{m+1} = \cdots = a_{n_k'-1} = 0.$$

Hence, if we put

$$s_n' = \frac{1}{2^n}\left\{s_0 + ns_1 + \frac{n(n-1)}{2!}s_2 + \cdots + s_n\right\},$$

then we have

$$\begin{aligned}
s_{2m}' - s_m &= \frac{1}{2^{2m}}\left\{s_0 + 2ms_1 + \frac{2m(2m-1)}{2!}s_2 + \cdots + s_{2m}\right\} \\
&\quad - \frac{1}{2^{2m}}\left\{s_m + 2ms_m + \frac{2m(2m-1)}{2!}s_m + \cdots + s_m\right\} \\
&= \frac{1}{2^{2m}}\left\{-(a_0 + \cdots + a_{n_k}) - 2m(a_1 + \cdots + a_{n_k})\right. \\
&\quad - \frac{2m(2m-1)}{2!}(a_2 + \cdots + a_{n_k}) - \cdots \\
&\quad - \frac{2m(2m-1)\cdots(2m-n_k+1)}{n_k!}a_{n_k} \\
&\quad + \frac{2m(2m-1)\cdots(2m-n_k')}{(n_k'+1)!}a_{n_k} + \cdots \\
&\quad \left. + (a_{n_k} + \cdots + a_{2m-1})\right\}.
\end{aligned}$$

Since from (2) we can find a positive constant M such that $|a_n| < Me^{n\log\alpha}$, $(n = 0, 1, 2, \cdots)$, we get

$$\begin{aligned}
|s_{2m}' - s_m| &< 2M\frac{e^{2m\log\alpha}}{2^{2m}}(n_k+1)^2\frac{2m(2m-1)\cdots(2m-n_k+1)}{n_k!} \\
&< 4M\frac{e^{2m\log\alpha}}{2^{2m}}\frac{2m\Gamma(2m)}{\Gamma(m+\lambda)\Gamma(m-\lambda)},
\end{aligned}$$

where $\lambda = m - n_k$.

Let us now put

$$f(m) = \frac{e^{2m\log\alpha}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m-\lambda)\Gamma(m+\lambda)},$$

or

$$f(m) = \frac{e^{2m\log\alpha}}{2^{2m}} \frac{2m\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))},$$

where $\lambda = m\delta$, $(0 < \delta = (n_k' - n_k + 1)/(n_k' + n_k + 1) < 1)$. Then

$$\begin{aligned}
\log f(m) &= 2m\log\alpha - 2m\log 2 + \log(2m) \\
&\quad + (2m - \tfrac{1}{2})\log(2m) - 2m + O(1) \\
&\quad - \{m(1-\delta) - \tfrac{1}{2}\}\log((1-\delta)m) + (1-\delta)m + O(1) \\
&\quad - \{m(1+\delta) - \tfrac{1}{2}\}\log((1+\delta)m) + (1+\delta)m + O(1) \\
&= -m\phi(\delta) + \tfrac{1}{2}\log m + O(1),
\end{aligned}$$

where

$$\phi(\delta) = (1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) - 2\log\alpha.$$

For a fixed number η_0, $(1 > \eta_0 \geqq 0)$, such that $\phi(\eta_0) = 0$, any number η such that $1 > \eta > \eta_0$ gives $\phi(\eta) > 0$. When η is so fixed, it follows from (1) that

$$\lim s_m = \lim s_{2m}' = s \quad \text{for} \quad 1 > \delta > \eta.$$

On the other hand, from $n_k'/n_k \geqq (1+\eta)/(1-\eta)$ we have

$$\frac{n_k' - n_k + 1}{n_k' + n_k + 1} > \eta.$$

Consequently $1 > \delta > \eta$ since

$$\delta = (n_k' - n_k + 1)/(n_k' + n_k + 1).$$

Therefore

$$\lim_{k\to\infty} s_{n_k} = \lim_{m\to\infty} s_m = s.$$

Thus our theorem is completely proved.

REMARK. From Theorem I and Knopp's* theorem follows immediately Ostrowski's theorem:†

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series whose radius of convergence is* 1. *If*

$$a_\nu = 0 \quad \text{for} \quad n_k < \nu < n_k',$$

and

$$\frac{n_k'}{n_k} > 1+\theta, \qquad (k = 1, 2, \cdots),$$

θ *being a positive constant, then the partial sums* s_{nk} *of this series converge uniformly in a full neighbourhood of every regular point of the function of* (z) *on the unit circle.*

THEOREM 2. *Let* $\sum a_n$ *be a given series summable by Euler's method to the sum* s, *and for two given increasing sequences* $\{n_k\}$, $\{n_k'\}$, $(n_k<n_k')$, *of positive integers and for* $p\geqq -1$ *let*

$$a_\nu = 0 \quad \text{for} \quad n_k < \nu < n_k', \quad (k = 1, 2, \cdots),$$
$$a_n = O(n^p).$$

If

$$\frac{n_k'}{n_k} \geqq 1 + c\left(\frac{3}{2}+p\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

holds for any given number c *greater than* 2, *and for all sufficiently great integers* k, *then we have*

$$\lim_{k\to\infty}\sum_{\nu=0}^{n_k} a_\nu = s.$$

PROOF. From the assumption we can take c' and q such that

$$\frac{n_k'-n_k}{n_k} \geqq c'\left(\frac{3}{2}+p+q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}, \qquad (c'>2, q>0).$$

Therefore

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* Knopp, Mathematische Zeitschrift, vol. 15 (1922).

† Zygmund (Journal of the London Mathematical Society, vol. 6 (1931)) proved Ostrowski's theorem similarly for the Borel method of summation.

$$\frac{n_k' - n_k}{n_k' + n_k} \geqq c'\left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

$$:\left(c'\left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2} + 2\right)$$

$$\geqq \left(\frac{3}{2} + p + q\right)^{1/2}\left(\frac{\log n_k}{n_k}\right)^{1/2}$$

for all sufficiently great integers k. Hence from

$$\delta = \frac{n_k' - n_k + 1}{n_k' + n_k + 1} > \frac{n_k' - n_k}{n_k' + n_k}$$

we get

$$\delta^2 \geqq \left(\frac{3}{2} + p + q\right)\frac{\log m}{m}, \qquad \left(m = \frac{n_k' + n_k + 1}{2}\right).$$

Consequently, from

$$(1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) > \delta^2,$$

we have

$$(1+\delta)\log(1+\delta) + (1-\delta)\log(1-\delta) > \left(\frac{3}{2} + p + q\right)\frac{\log m}{m},$$

whence as in the proof of Theorem I we obtain

$$|s_{2m}' - s_m| < Mm\frac{(2m)^p}{2^{2m}}\frac{\Gamma(2m)}{\Gamma(m(1-\delta))\Gamma(m(1+\delta))}$$

$$< M' \exp\left[m\left\{\left(\frac{3}{2} + p\right)\frac{\log m}{m} - (1+\delta)\log(1+\delta)\right.\right.$$

$$\left.\left. - (1-\delta)\log(1-\delta)\right\} + O(1)\right]$$

$$\to 0, \quad (m \to \infty),$$

M, M' being constants. Therefore

$$\lim s_{n_k} = \lim s_m = s.$$

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NOTE ON THE CONTINUITY OF THE ERGODIC FUNCTION

BY M. H. MARTIN

1. *Introduction.* Let M denote a bounded point set and ϵ an arbitrarily chosen positive number. A continuous curve C is termed *ϵ-ergodic to M* if an arbitrary point of M lies at a distance $\leqq\epsilon$ from some point of C. Recently* I have shown that the set of continuous, rectifiable curves ϵ-ergodic to M contains a member whose length furnishes an absolute minimum for the lengths of the curves in the set. This member was called an *ergodic curve of M* and its length the *ergodic function $\Lambda(\epsilon)$ of M*. The function $\Lambda(\epsilon)$ is finite and non-negative, being equal to zero if and only if $\epsilon\geqq\rho$, where ρ is the radius of the smallest circular region containing M. In addition $\Lambda(\epsilon)$ was proved to be a monotone non-increasing function of ϵ which is always continuous on the right.

In this note it is shown that $\Lambda(\epsilon)$ is also continuous on the left (and is therefore continuous in the ordinary sense). In the original version of this paper I was able to prove this result only for a value $\epsilon_0(<\rho)$ of ϵ for which the set M had an ergodic curve which was an "ordinary curve" (a continuous curve which is either of class C' or else made up of a finite number of arcs of class C'). The general result announced above is made possible by Lemma 2 below for which I am indebted to Professor von Neumann.

2. *Preliminary Lemmas.* In this section we shall assemble a number of lemmas leading to the proof of the result announced in the introduction.

LEMMA 1. *The set M_1 of points lying at a distance $\leqq\epsilon$ from the points of a continuous rectifiable arc of length $2s$ joining two points A and B situated a distance $2c(c\leqq s)$ apart lies in a region composed of the points interior to two circles described about A and B as centers with radii equal to $\epsilon+(2^{1/2}\alpha+s/(2\epsilon))s$, where $\alpha^2=1-c/s$.*

* *Ergodic curves*, American Journal of Mathematics, vol. 58 (1936), pp. 727–734.

For the proof of this lemma we first observe that a point on the rectifiable arc either lies in the interior of an ellipse E having A and B as foci and a major axis of length $2s$, or lies on E itself. The proof of this fact is elementary and is omitted.

The set E_ϵ of points lying at a distance $\leqq\epsilon$ from either the points of E or the points interior to E therefore contains the set M_1.

The boundary of E_ϵ is a closed analytic curve forming part of the envelope of the family of circles of radius ϵ with centers on E.

E_ϵ is divided into two halves by the minor axis of E produced. Let us consider that half which contains the focus A. An elementary calculation shows that an absolute maximum R of the distances of the points of this half of E_ϵ from A is furnished by the two points in which the minor axis of E produced intersects the boundary of E_ϵ. Upon computing this distance, we find

$$R = [\epsilon^2 + 2\epsilon(s^2 - c^2)^{1/2} + s^2]^{1/2}.$$

Since $c \leqq s$ and $s - c = \alpha^2 s$, it follows that

$$R \leqq [\epsilon^2 + 2^{3/2}\epsilon\alpha s + s^2]^{1/2} < \epsilon + \left(2^{1/2}\alpha + \frac{s}{2\epsilon}\right)s.$$

A circle of radius $\epsilon + (2^{1/2}\alpha + s/(2\epsilon))s$ described about A as a center accordingly contains in its interior the half of E_ϵ containing A. A similar circle described about B as a center will contain the remaining half of E_ϵ in its interior. The two circles taken together will then contain the whole of E_ϵ and will therefore *a fortiori* contain the set M_1 in their interiors.

LEMMA 2. *Let*

$$C\colon\ x = x(s), \quad y = y(s), \qquad 0 \leqq s \leqq L, \qquad (s = \text{arc length}),$$

be an open continuous rectifiable curve of length L. Let C be divided into n arcs of lengths $s_1^{(n)}, s_2^{(n)}, \cdots, s_n^{(n)}$ subtending chords of lengths $c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)}$ respectively. Define the n quantities $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_n^{(n)}$ by the equations

$$\alpha_i^{(n)} = \left[1 - \frac{c_i^{(n)}}{s_i^{(n)}}\right]^{1/2}, \qquad (i = 1, 2, \cdots, n),$$

and then the $n+1$ quantities $\bar{\alpha}_0^{(n)}, \bar{\alpha}_1^{(n)}, \cdots, \bar{\alpha}_n^{(n)}$ by the relations

$$\bar{\alpha}_0^{(n)} = \alpha_1^{(n)}, \quad \bar{\alpha}_i^{(n)} = \alpha_i^{(n)} + \alpha_{i+1}^{(n)} \quad \textit{for} \quad i = 1, 2, \cdots, n-1,$$
$$\bar{\alpha}_n^{(n)} = \alpha_n^{(n)}.$$

If $s_1^{(n)} = s_2^{(n)} = \cdots = s_n^{(n)} = L/n$, *the arithmetic mean*

$$\bar{\bar{\alpha}}_n = \frac{1}{n+1} \sum_{i=0}^{n} \bar{\alpha}_i^{(n)}$$

of $\bar{\alpha}_0^{(n)}, \bar{\alpha}_1^{(n)}, \cdots, \bar{\alpha}_n^{(n)}$ *tends to zero as* n *tends to infinity.*

We have

$$\bar{\bar{\alpha}}_n = \frac{2}{n+1} \sum_{i=1}^{n} \alpha_i^{(n)} < \frac{2}{n} \sum_{i=1}^{n} \alpha_i^{(n)},$$

and therefore, upon substituting the values of $\alpha_i^{(n)}$ given above and applying Schwarz's inequality.

$$\bar{\bar{\alpha}}_n < \frac{2}{n} \left[n \sum_{i=1}^{n} \left(1 - \frac{c_i^{(n)}}{s_i^{(n)}} \right) \right]^{1/2},$$

whereupon, since $s_i^{(n)} = L/n$, it follows that

$$\bar{\bar{\alpha}}_n < 2 \left[1 - \frac{\sum_{i=1}^{n} c_i^{(n)}}{L} \right]^{1/2}.$$

Now C is rectifiable of length L. Therefore $\lim_{n \to \infty} \sum_{i=1}^{n} c_i^{(n)} = L$, and consequently $\lim_{n \to \infty} \bar{\bar{\alpha}}_n = 0$.

LEMMA 3. *Let*

$$C: \quad x = x(s), \qquad y = y(s), \qquad 0 \leqq s \leqq L,$$

be a continuous rectifiable curve of length L. *Denote by* C_ϵ *the set of points lying at a distance* $\leqq \epsilon$ *from the points of* C. *There exists a sequence* $\{C_n\}$ *of continuous rectifiable curves for which, if* L_n *denotes the length of* C_n, *the following hold*:

(a) $\lim L_n = L$,

(b) C_n *is* ϵ_n*-ergodic to* C_ϵ *with* $\epsilon_n < \epsilon$ *and* $\lim \epsilon_n = \epsilon$.

In proving this lemma it is convenient to take C to be an open curve, the proof permitting an immediate adaptation to the case where C is closed. Divide C up into n equal arcs each

of length L/n by inserting $n+1$ points on C. These $n+1$ points we shall denote by 0, 1, 2, $\cdots$, n, where 0 and n denote the endpoints of C.

About a point i, $(i=0, 1, 2, \cdots, n)$, of the subdivision we describe two circles, one, Γ_i, of radius $\epsilon+(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n)$, the other, γ_i, of radius $(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/n$, where in each case $\bar{\alpha}_i^{(n)}$ denotes the quantity defined in Lemma 2.

The totality of points interior to the circles Γ_i constitutes a point set which we designate by M_n. On placing $s=L/(2n)$ in Lemma 1, we ascertain that $C_\epsilon \subset M_n$.

A point making a complete circuit of a circle γ_i comes within a distance $\leqq(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/n$ of the points enclosed by γ_i, and within a distance $\leqq\epsilon-(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n)$ of those points outside γ_i but inside Γ_i. For sufficiently great values of n the latter of the two distances is the greater. Consequently, for sufficiently great values of n, a point making a complete circuit of γ_i comes within a distance $\leqq\epsilon-(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n)\leqq\epsilon-L^2/(8n^2\epsilon)$ of every point interior to Γ_i.

Now consider a point P which traverses C from one endpoint to the other and which makes complete circuits of all the circles γ_i during the process. More precisely, P moves as follows: starting at the endpoint 0 of C, the point P moves along C in the direction of increasing arc length until it meets the circle γ_0 with center 0 for the first time; P then makes a complete circuit of γ_0; after returning to the curve C, the point P continues along C in the direction of increasing arc length until it meets the circle γ_1 with center 1 for the first time; P then makes a complete circuit of γ_1; $\cdots$. At the completion of this process P coincides with n and will have traced out a continuous rectifiable curve C_n of length L_n, where

$$L_n = L + \sum_{i=0}^{n} 2\pi\left(2^{1/2}\bar{\alpha}_i^{(n)} + \frac{L}{4n\epsilon}\right)\frac{L}{n}.$$

On replacing $\sum_{i=0}^{n}\bar{\alpha}_i^{(n)}$ by its value $(n+1)\bar{\alpha}_n$ (introduced in Lemma 2) and carrying out the remainder of the summation, we find

$$L_n = L\left[1 + 2\pi\frac{n+1}{n}\left(2^{1/2}\bar{\alpha}_n + \frac{L}{4n\epsilon}\right)\right].$$

In addition the curve C_n is ϵ_n-ergodic to M_n, where

$$\epsilon_n = \epsilon - \frac{L^2}{8n^2\epsilon} < \epsilon.$$

The proof of the lemma is now completed by observing that, since $C_\epsilon \subset M_n$, the curve C_n is ϵ_n-ergodic to C_ϵ, and that

$$\lim \epsilon_n = \epsilon, \qquad \lim L_n = L,$$

the latter of which follows from Lemma 2.

3. *The Continuity of* $\Lambda(\epsilon)$. Let ϵ_0 denote a fixed value of ϵ lying in the interval $0<\epsilon<\rho$ (for the definition of ρ see the introduction). Let C be an ergodic curve of M for $\epsilon=\epsilon_0$. Write $\Lambda(\epsilon_0)=L$, so that the length of C is L. Let C_{ϵ_0} denote the set of points lying at a distance $\leqq\epsilon_0$ from points of C. We note that $M \subset C_{\epsilon_0}$. The ergodic function $\Lambda(\epsilon)$ will now be demonstrated to be continuous on the left for $\epsilon=\epsilon_0$.

If $\Lambda(\epsilon)$ be discontinuous on the left for $\epsilon=\epsilon_0$, we shall have, since $\Lambda(\epsilon)$ is monotone non-increasing

$$\lim \Lambda(\epsilon_n) = L + \delta, \qquad (\delta > 0),$$

for an arbitrarily selected sequence $\{\epsilon_n\}$ of ϵ values for which

$$\epsilon_n < \epsilon_0, \qquad \lim \epsilon_n = \epsilon_0.$$

On the other hand, according to Lemma 3 there exists a sequence $\{C_n\}$ of continuous rectifiable curves such that, if L_n denotes the length of C_n, we have

(a) $\lim L_n = L$,

(b) C_n is ϵ_n-ergodic to C_{ϵ_0} (and therefore ϵ_n-ergodic to M) with $\epsilon_n<\epsilon_0$ and $\lim \epsilon_n=\epsilon_0$.

Therefore, since $\Lambda(\epsilon_n)\leqq L_n$, it follows that

$$\lim \Lambda(\epsilon_n) \leqq L,$$

which is a contradiction to our previous result. Hence $\Lambda(\epsilon)$ is continuous on the left for $\epsilon=\epsilon_0$.

The case $\epsilon_0<\rho$ is now disposed of. There remains the case $\epsilon_0=\rho$ to consider. That $\Lambda(\epsilon)$ is continuous on the left in this case is trivial. The ergodic curve C shrinks to a point P, since $\Lambda(\rho)=0$, and one replaces the curve L_n above by a circle of radius $1/n$ described about P as center.

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REAL CANONICAL BINARY SYMMETRIC TRILINEAR FORMS*

BY RUFUS OLDENBURGER

1. *Introduction.* In treating the equivalence of binary forms of the third degree under non-singular linear transformations it is natural to divide the problem into three parts. These are concerned respectively with the equivalence of forms of the type

$$a_{ijk}x_iy_jz_k,$$

where

(a) the vectors (x_1, x_2), (y_1, y_2), (z_1, z_2) are distinct;

(b) 2 of the vectors (x_1, x_2), (y_1, y_2), (z_1, z_2) are equal but not equal to the 3rd;

(c) the vectors (x_1, x_2), (y_1, y_2), (z_1, z_2) are equal.

Dedekind, Schwartz, others,† and the author have treated the problem of the equivalence of binary trilinear forms in the complex field. Specifically the problem solved was that of determining the conditions on (a_{ijk}), (a'_{pqr}) such that the form $T=a_{ijk}x_iy_jz_k$ be reducible to the given form $T'=a'_{pqr}x'_py'_qz'_r$ under the linear transformations

$$(1) \qquad x_i = a_{ip}x'_p, \qquad y_j = b_{jq}y'_q, \qquad z_k = c_{kr}z'_r,$$

where $i, j, k, p, q, r=1, 2$, and the matrices (a_{ip}), (b_{jq}), (c_{kr}) are non-singular. Recently the author solved this problem of equivalence for the field of reals.‡ The theory for case (a) in the field of reals and complexes is hence complete. In the present paper the author treats the equivalence of forms satisfying (b) for the field of reals.

Evidently, the *class of forms* (c) *is contained in the class* (b), which in turn is contained in the class (a).

In a new paper, to appear elsewhere, the author has solved the problem of equivalence of the forms in class (c), making

* Presented to the Society, March 26, 1937.

† For a complete list of references see R. Oldenburger, *On canonical binary trilinear forms*, this Bulletin, vol. 38 (1932), p. 385.

‡ R. Oldenburger, *Real canonical binary trilinear forms*, American Journal of Mathematics, vol. 59 (1937), pp. 427–435.

use of the results obtained here for class (b), many of which also hold for class (c). *Class* (c) *is simply the class of all binary cubic forms* $a_{ijk}x_ix_jx_k$. The theory of forms of classes (a), (b), (c) is thus complete for the field of reals.

The forms in class (b) with real coefficients can always be written as

$$T = a_{ijk}x_iy_jy_k$$

where the matrix $A=(a_{ijk})$ is symmetric with respect to the j and k indices; $a_{112}=a_{121}$, $a_{123}=a_{132}$, $\cdots$. The matrix A is then unique. It is proved in the present paper that T is always equivalent under transformations (1) on the x's and y's to one of six canonical forms. These six canonical forms correspond to six canonical pairs of quadratic equations to which pairs of quadratic equations with no linear terms are equivalent.

The treatment involves the use of arithmetic rank invariants, and properties of invariant factors. The technique is essentially different from that used in earlier papers on binary trilinear forms.

2. *Definitions.* The *rank* r_i of a 3-way matrix $A=(a_{ijk})$, $(i, j, k=1, 2)$, and its associated trilinear form $T=a_{ijk}x_iy_jz_k$ is the rank of the 2-way matrix

$$\begin{pmatrix} a_{111} & a_{112} & a_{121} & a_{122} \\ a_{211} & a_{212} & a_{221} & a_{222} \end{pmatrix}.$$

Assume that a_{ijk}, $(i, j, k=1, 2)$, are not all zero. The 3-way rank $r[jk, i]$ of A and F is defined to be 2 or 1 according as the quantities

$$(2) \qquad \begin{vmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{vmatrix}, \begin{vmatrix} a_{211} & a_{212} \\ a_{221} & a_{222} \end{vmatrix}, \begin{vmatrix} a_{111} & a_{212} \\ a_{121} & a_{222} \end{vmatrix} + \begin{vmatrix} a_{211} & a_{112} \\ a_{221} & a_{122} \end{vmatrix}$$

are not all zero or are all zero. These are 3-way determinants with i non-signant.

These ranks have been proved to be invariant under non-singular linear transformations* on T.

3. *The Case* $r_i=r_j=r_k=2$. We shall need the following lemma:

* R. Oldenburger, *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), pp. 622–657.

LEMMA 1. *Let A, B be symmetric 2-way matrices of the second order. If $|\rho A+\sigma B| \not\equiv 0$ and A, B are linearly independent, the quadratic form $|\rho A+\sigma B|$ is either indefinite or negative definite.*

Under transformation of A, B to

$$X'AX = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad X'BX = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where X is non-singular, the determinant $|\rho A+\sigma B|$ goes into $|X|^2\,|\rho A+\sigma B|$. If $\alpha\beta=0$, then $|\rho A+\sigma B|$ contains the factor σ. If the determinant of the symmetric matrix of the quadratic form $|\rho A+\sigma B|$ is zero, $|\rho A+\sigma B|$ is equivalent* under non-singular linear transformation on ρ, σ to $\alpha\rho'^2$, whence $|\rho A+\sigma B|$ can be written in the factored form $\alpha(g\rho+h\sigma)^2$. Hence, if $|\rho A+\sigma B|$ does not factor, $\alpha\beta\neq 0$ and the determinant of the form $|\rho A+\sigma B|$ is $\neq 0$. The matrix of $|\rho A+\sigma B|$ is then regular and non-singular. Then $|\rho A+\sigma B|$ is equivalent to $g\rho^2+h\sigma^2$. If g, h are of unlike sign the determinant of $|\rho A+\sigma B|$ is <0, and $|\rho A+\sigma B|$ factors. If g, h have the same sign this determinant is >0. Hence if $|\rho A+\sigma B|$ does not factor, the determinant of $|\rho A+\sigma B|$ is >0, whence

$$(\alpha c - \beta a)^2 + 4b^2\alpha\beta < 0.$$

If $\alpha\beta>0$ this is not satisfied. Hence if $|\rho A+\sigma B|$ does not factor, $\alpha\beta<0$ and the determinant of $|\rho A+\sigma B|$ is >0, whence $|\rho A+\sigma B|$ is negative definite.

If the determinant of $|\rho A+\sigma B|$ is zero, then

$$(\alpha c - \beta a)^2 + 4b^2\alpha\beta = 0. \tag{2'}$$

If $\alpha\beta>0$, (2') cannot be satisfied unless $b=0$ and $\alpha c-\beta a=0$. Then A, B are linearly dependent. If $\alpha\beta=0$, (2') implies that $\alpha c-\beta a=0$, whence, since α, β are not both zero, the product $ac=0$. This implies $\alpha c=\beta a=0$. Hence $|\rho A+\sigma B| = -b^2\sigma^2$, which is negative definite if $b\neq 0$. As before $b=0$ implies the linear dependence of A, B. If $\alpha\beta<0$ the matrix of the form $|\rho A+\sigma B|$ is regular whence $|\rho A+\sigma B|$ is negative definite. If the determinant of $|\rho A+\sigma B|$ is $\neq 0$, and $\alpha\beta=0$, then $|\rho A+\sigma B|$ is indefinite, or is $-b^2\sigma^2$, which is negative definite.

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* L. E. Dickson, *Modern Algebraic Theories*, p. 88.

If the rank $r[jk, i]$ of the symmetric form $T=a_{ijk}x_iy_jy_k$, $(i, j, k=1, 2)$, is 1, then* $r_j=r_k=1$. Hence $r[jk, i]=2$. Let $A=(a_{1jk})$, $B=(a_{2jk})$. The matrices A, B are symmetric matrices. Since the coefficients of ρ^2, $\rho\sigma$, σ^2 in the quadratic form $|\rho A+\sigma B|$ are the quantities (2), the determinant $|\rho A+\sigma B|\not\equiv 0$. Since $r_i=2$, the matrices A, B are linearly independent. The matrices A, B hence satisfy the conditions of Lemma 1.

If the quadratic form $|\rho A+\sigma B|$ is indefinite, it is equivalent under non-singular linear transformations† on ρ, σ to $\rho'^2-\sigma'^2$ which is further equivalent to $\rho''\sigma''$ under the transformations $\rho''=\rho'-\sigma'$, $\sigma''=\rho'+\sigma'$. These transformations correspond‡ to non-singular linear transformations on x_1, x_2 of T, to give a new form $T'=a'_{ijk}x'_iy_jy_k$, for which $|\rho''A'+\sigma''B'|=\rho''\sigma''$, where $A'=(a'_{1jk})$, $B'=(a'_{2jk})$. It follows that A', B' are singular. The forms $T_1'=a'_{1jk}y_jy_k$, $T_2'=a'_{2jk}y_jy_k$ are equivalent§ to a pair $T_1''=\alpha y_1'^2$, $T_2''=l(y_1', y_2')\cdot m(y_1', y_2')$, where l, m are linear in y_1', y_2'. Now $lm\not\equiv ky_1'^2$, since otherwise $r_i<2$. It is hence no restriction to take $l\not\equiv ky_1'$. Let $y_1''=y_1'$, $y_2''=l(y_1', y_2')$, whence the pair T_1'', T_2'' transforms non-singularly into $T_1'''=\alpha y_1''^2$, $T_2'''=y_2''n(y_1'', y_2'')$, where n is linear in y_1'', y_2''. Since T_2''' is singular, $n(y_1'', y_2'')=\beta y_2''$ for some β. Since $r_i=2$; $\alpha, \beta\neq 0$. Let $x_1''=\alpha x_1'$, $x_2''=\beta x_2'$. Then $x_1'T_1'''+x_2'T_2'''$ transforms into the *canonical form*

$$R = x_1y_1^2 + x_2y_2^2,$$

where the primes on the variables have been dropped for the sake of simplicity.

If $|\rho A+\sigma B|$ is a negative definite quadratic form of rank 1, it is equivalent‖ to $-\sigma'^2$. This corresponds to the transition from T to $T'=a'_{ijk}x'_iy_jy_k$, where $|a_{1jk}|=0$, and

$$\begin{vmatrix} a'_{111} & a'_{212} \\ a'_{121} & a'_{222} \end{vmatrix} + \begin{vmatrix} a'_{211} & a'_{112} \\ a'_{221} & a'_{122} \end{vmatrix} = 0. \tag{3}$$

* R. Oldenburger, op. cit., p. 649.

† L. E. Dickson, *Modern Algebraic Theories*, p. 72.

‡ R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, Transactions of this Society, vol. 39 (1936), p. 432.

§ L. E. Dickson, op. cit., p. 69.

‖ L. E. Dickson, op. cit., p. 72.

The pair of quadratic forms $a'_{ijk}y_iy_k$, $a'_{2jk}y_iy_k$ is now equivalent to a pair $\alpha y_1'^2$, $ay_1'^2+2by_1'y_2'+cy_2'^2$, for which

$$\left|\rho'\begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix}+\sigma'\begin{pmatrix}a & b\\ b & c\end{pmatrix}\right| = -K^2\sigma'^2,$$

where K is the modulus of transformation. The analogue of (3) gives $\alpha c=0$, whence, since $\alpha\neq 0$, we have $c=0$. Now $b\neq 0$, since $r_i=2$. If we write

$$x_1'' = \alpha x_1', \quad x_2'' = x_2', \quad y_1'' = y_1', \quad y_2'' = \frac{a}{2}y_1' + by_2',$$

the form $x_1'\alpha y_1'^2+x_2'(ay_1'^2+2by_1'y_2')$ transforms into the *canonical form*

$$L = x_1y_1^2 + 2x_2y_1y_2,$$

where the primes have been dropped.

To treat the class of forms T for which $|\rho A+\sigma B|$ does not factor in the field of reals we shall need another lemma:

LEMMA 2. *A pair P of quadratic forms x^2-y^2, $ex^2+2fxy+gy^2$ is equivalent under non-singular linear transformations to the pair x^2-y^2, $2xy$ if and only if the characteristic determinant*

$$\left|\rho\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\sigma\begin{pmatrix}e & f\\ f & g\end{pmatrix}\right|$$

of P is $-\rho^2-\sigma^2$.

The necessity is evident.* To prove the sufficiency assume that the characteristic determinant of P is $-\rho^2-\sigma^2$. Then e, f, g satisfy the relations

$$g = e, \qquad f^2 - e^2 = 1. \tag{4}$$

An elementary computation reveals that the matrix X of the most general non-singular linear transformation leaving x^2-y^2 invariant in the sense that

$$X'\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}X = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

where X' is X-transpose, is of the form

* L. E. Dickson, op. cit., p. 97.

$$X = J\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$a^2 - b^2 = 1.$$

If the pairs of forms of the theorem are equivalent there exists a matrix X of the above form such that

$$X'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X = E,$$

where

$$E = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

Since

$$J'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it follows that

$$X'\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \equiv \pm \begin{pmatrix} 2ab & a^2 + b^2 \\ a^2 + b^2 & 2ab \end{pmatrix}. \tag{5}$$

We shall show that it is always possible to choose the matrix (5) equal to E, where the elements of E satisfy (4). Now E is of the form

$$E_1 = \begin{pmatrix} e & (1 + e^2)^{1/2} \\ (1 + e^2)^{1/2} & e \end{pmatrix},$$

or

$$E_2 = \begin{pmatrix} e & -(1 + e^2)^{1/2} \\ -(1 + e^2)^{1/2} & e \end{pmatrix}.$$

Take $e \neq 0$. If $E = E_1$ equate E_1 and (5) with the $+$ sign: if $E = E_2$ equate E_2 and (5) with the $-$ sign. We obtain

$$(6) \qquad e = \pm 2ab, \qquad (1 + e^2)^{1/2} = a^2 + b^2.$$

We obtain as real solutions of (6)

$$b = \left(\frac{(1 + e^2)^{1/2} - 1}{2}\right)^{1/2}; \qquad a = \pm \frac{e}{2b};$$

which identically satisfy the condition $a^2 - b^2 = 1$. If $e = 0$,

$$E_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_2 \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In the latter case take the $-$ sign in (5), and $a = 0$, $b = 1$. Then (5) $\equiv E_2$.

We shall now show by Lemmas 1 and 2 that $T = a_{ijk}x_iy_jy_k$, where $|\rho A + \sigma B|$ does not factor, is equivalent to a *canonical form* of the type

$$M = x_1(y_1^2 - y_2^2) + 2x_2y_1y_2.$$

By Lemma 1 $|\rho(a_{1jk}) + \sigma(a_{2jk})|$ is equivalent under non-singular linear transformations on ρ, σ to $-\rho^2 - \sigma^2$, since it is negative definite. This corresponds to a non-singular linear transformation* on x_1, x_2 of T to give a new form $T' = a'_{ijk}x'_iy_jy_k$ such that

$$|\rho(a'_{1jk}) + \sigma(a'_{2jk})| = -\rho^2 - \sigma^2.$$

Since $|a'_{1jk}| = -1$, the form $a'_{1jk}y_jy_k$ is equivalent under orthogonal transformations† on the y's to $\alpha y_1'^2 - y_2'^2/\alpha$, $\alpha > 0$, and $a'_{2jk}y_jy_k$ transforms simultaneously to a new form T'_2. This pair of forms is further equivalent under the transformations $y_1'' = \alpha^{1/2}y_1'$, $y_2'' = y_2'/\alpha^{1/2}$ of determinant unity to a pair $C_3 = y_1''^2 - y_2''^2$, T_2'' with matrices which will be denoted by A', B' respectively. Now $|\rho A' + \sigma B'| = -\rho^2 - \sigma^2$. By Lemma 2, $x_1'C_3 + x_2'T_2''$ is equivalent to M.

Since the rank and index of the quadratic form $|\rho(a_{1jk}) + (a_{2jk})|$ are invariant under non-singular linear transformations on $T = a_{ijk}x_iy_jy_k$, no two in the set of canonical forms R, L, M are equivalent.

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* See R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, loc. cit.

† L. E. Dickson, op. cit., p. 74.

4. *The Cases where* $(r_i, r_j, r_k) \neq (2, 2, 2)$. By symmetry $r_j = r_k$. If $r_i = 2$, $r_j = r_k = 1$, then* $r[ij, k] = r[kj, i] = r[ik, j] = 1$, which, as is easy to show, implies that $r_i = 1$, giving a contradiction. If $r_i = 1$, $r_j = r_k = 2$, then $T = a_{ijk}x_iy_jy_k$ is *equivalent* at once to $P = x_1(y_1^2 + y_2^2)$ or $Q = x_2(y_1^2 - y_2^2)$. In the first case $|\rho(a_{1jk}) + \sigma(a_{2jk})|$ is positive definite, in the latter case negative definite. If $r_i = r_j = r_k = 1$, T is equivalent to $x_1y_1^2$. Evidently P, Q are not equivalent.

5. *Results.* By the invariance of rank it follows that the canonical forms obtained are not equivalent. We have proved this theorem:

THEOREM. *Two binary symmetric forms* $T = a_{ijk}x_iy_jy_k$ *and* $T' = a'_{ijk}x'_iy'_jy'_k$ *are equivalent if and only if they have the same set of ranks* $[r_i, r_j, r_k]$, *and the quadratic forms* $|\rho(a_{1jk}) + \sigma(a_{2jk})|$, $|\rho(a'_{1jk}) + \sigma(a'_{2jk})|$ *have the same rank and index.*

It follows from the theory that a pair of quadratic equations with no linear terms is equivalent to one of the canonical pairs

$$[x^2 = a, y^2 = b], \quad [x^2 = a, xy = b], \quad [x^2 - y^2 = a, xy = b],$$
$$[x^2 + y^2 = a, 0 = 0], \quad [x^2 - y^2 = a, 0 = 0],$$

or $[x^2 = a, 0 = 0]$, where the roots of the original pair are linear in the roots of the canonical pair.

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* R. Oldenburger, *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), p. 641.

SOME INEQUALITIES CONCERNING FUNCTIONS OF EXPONENTIAL TYPE

BY R. J. DUFFIN AND A. C. SCHAEFFER

In this paper we derive two inequalities concerning entire functions satisfying the conditions $f(z)=O(e^{n|z|})$, and $|f(z)| \leqq 1$ on the real axis. Our results are closely related to theorems given by S. Bernstein, Szegö, van der Corput and Schaake, and Boas.*

THEOREM I. *Let $f(z)$ be an entire function such that*

$$f(z) = O(e^{n|z|}) \tag{1}$$

uniformly over the entire plane, and on the real axis

$$|f(z)| \leqq 1. \tag{2}$$

Then, if a and b are any two real numbers,

$$|af(z) + bf'(z)| \leqq (a^2 + n^2b^2)^{1/2} \tag{3}$$

for real z.

PROOF OF THEOREM I: By means of Cauchy's Integral Formula and the Phragmén-Lindelöf Principle it is not difficult to show that if $f(z)$ satisfies the conditions of Theorem I then

$$\frac{d}{dz}\left\{\frac{f(z)}{\sin n(z+\alpha)}\right\} = -n\sum_{k=-\infty}^{\infty} \frac{(-1)^k f\left(\frac{k\pi}{n}-\alpha\right)}{\{n(z+\alpha)-k\pi\}^2}. \tag{4}$$

For the proof of (4) see Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. II, page 35 and page 218. If α is real we have $|f(k\pi/n-\alpha)| \leqq 1$, hence for all real z and α

$$|f'(z)\sin n(z+\alpha) - nf(z)\cos n(z+\alpha)| \leqq \sum_{-\infty}^{\infty} \frac{n\sin^2 n(z+\alpha)}{\{n(z+\alpha)-k\pi\}^2}. \tag{5}$$

* S. Bernstein, Comptes Rendus, vol. 176 (1923), p. 1603.

Van der Corput and Schaake, Compositio Matematica, vol. 2 (1935), p. 321; vol. 3 (1936), p. 128.

Szegö, Schriften der Königsberger gelehrten Gesellschaft, Naturwissenschaftliche Klasse, Fünftes Jahr 4 (1928), p. 69.

Boas, Transactions of this Society, vol. 40 (1936), p. 287.

Using the expansion $1/\sin^2 \pi w = \sum_{-\infty}^{\infty} 1/(nw - k\pi)^2$, we obtain

$$|f'(z) \sin n(z+\alpha) - nf(z) \cos n(z+\alpha)| \leqq n. \tag{6}$$

Given any real numbers a and b we may choose an α such that $\sin n(z+\alpha) = b/(a^2/n^2+b^2)^{1/2}$ and $-n \cos n(z+\alpha) = a/(a^2/n^2+b^2)^{1/2}$. Substitution in (6) then proves Theorem I.

THEOREM II. *Let $f(z)$ be a function which is real on the real axis and which satisfies the conditions of Theorem* I. *Then, if $z = x+iy$,*

$$|f'(z)|^2 + n^2|f(z)|^2 \leqq n^2 \cosh 2ny. \tag{7}$$

Unless $f(z)$ is of the form $\cos n(z+\alpha)$ the equality sign can occur only at points on the real axis where $f = \pm 1$.

PROOF OF THEOREM II: If $n(z+\alpha)$ is not a multiple of π the equality sign can occur in (5) only if $\pm f(k\pi/n - \alpha) = (-1)^k$ for all k. Putting these values for $f(k\pi/n-\alpha)$ in (4) one finds $f(z) \equiv \pm \cos n(z+\alpha)$. If $n(z+\alpha)$ is a multiple of π the equality sign in (5) clearly can occur only if $f(z) = \pm 1$.

In (3) let $a = n^2 f$ and $b = f'$. Then we find for real z

$$\{f'(z)\}^2 + n^2\{f(z)\}^2 \leqq n^2. \tag{8}$$

We shall suppose throughout that $f(z)$ is not of the form $\cos n(z+\alpha)$ for any real α. Then the equality sign can occur in (8) only at points where $f(z) = \pm 1$. From (8) we have $|f'(z)| < n$ for real z. It is easy to show that $f'(z) = O(e^{n|z|})$, so that $f'(z)/n$ satisfies all the conditions that $f(z)$ does, and we have from (8)

$$\left\{\frac{f''(z)}{n}\right\}^2 + n^2\left\{\frac{f'(z)}{n}\right\}^2 < n^2. \tag{9}$$

This is a strict inequality, for the equality can occur only at points where $f'(z)/n = \pm 1$, and we have shown that $|f'(z)| < n$.

In the same way one shows by induction that for the higher derivatives

$$\left\{\frac{f^{(k+1)}(z)}{n^k}\right\}^2 + n^2\left\{\frac{f^{(k)}(z)}{n^k}\right\}^2 < n^2.$$

Thus if $f(z)$ is expanded in a power series

$$f(z) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{\nu!} z^\nu,$$

we have $n^2a_\nu^2+a_{\nu+1}^2\leqq n^{2(\nu+1)}$, or

$$|na_\nu+ia_{\nu+1}|\leqq n^{\nu+1}. \tag{10}$$

The equality sign can occur only if $\nu=0$.

Let us write $f(z)$ in the form

$$f(z)=g(z)+u(z)=\sum_0^\infty \frac{a_{2\nu}}{(2\nu)!}z^{2\nu}+\sum_0^\infty \frac{a_{2\nu+1}}{(2\nu+1)!}z^{2\nu+1}$$

where $g(z)$ is even and $u(z)$ is odd. We see that on the imaginary axis ($z=iy$) the functions $g(z)$ and $u'(z)$ are real while $g'(z)$ and $u(z)$ are pure imaginary. Let $|y|>0$. Then

$$n^2|f(iy)|^2+|f'(iy)|^2=n^2|g(iy)|^2+n^2|u(iy)|^2 +|g'(iy)|^2+|u'(iy)|^2.$$

Combining the two even functions g and u' we obtain

$$n^2|g(iy)|^2+|u'(iy)|^2=|ng(iy)+iu'(iy)|^2 =\left|\sum_0^\infty \frac{na_{2\nu}+ia_{2\nu+1}}{(2\nu)!}(iy)^{2\nu}\right|^2.$$

By the inequality (10) this is less than

$$\left|\sum_0^\infty \frac{n^{2\nu+1}}{(2\nu)!}y^{2\nu}\right|^2=n^2\cosh^2 ny.$$

Combining the odd functions u and g', we have

$$n^2|u(iy)|^2+|g'(iy)|^2=|nu(iy)+ig'(iy)|^2 =\left|\sum_0^\infty \frac{na_{2\nu+1}+ia_{2\nu+2}}{(2\nu+1)!}(iy)^{2\nu+1}\right|^2$$

$$<\left|\sum_0^\infty \frac{n^{2\nu+2}}{(2\nu+1)!}y^{2\nu+1}\right|^2=n^2\sinh^2 ny.$$

Thus we have

$$n^2|f(iy)|^2+|f'(iy)|^2<n^2\cosh^2 ny+n^2\sinh^2 ny=n^2\cosh 2ny.$$

We have thus demonstrated that (7) holds on the imaginary axis; but if β is real $f(z+\beta)$ also satisfies the conditions, so this is sufficient. We remark that if $f(z)=\cos n(z+\alpha)$, (7) becomes an equality throughout the plane.

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NOTE ON THE RELATION BETWEEN CONTINUITY AND DEGREE OF POLYNOMIAL APPROXIMATION IN THE COMPLEX DOMAIN*

BY J. L. WALSH AND W. E. SEWELL

1. *Introduction.* It is the purpose of the present note to establish the following theorems:

THEOREM I. *Let C be an analytic Jordan curve in the z-plane and let $f(z)$ be defined in $\overline{C}$, the closed limited point set bounded by C. For each n, $n=1, 2, \cdots$, let a polynomial $P_n(z)$ of degree n in z exist such that*

$$(1) \qquad |f(z) - P_n(z)| \leqq \frac{M}{n^{p+\alpha}}, \qquad z \text{ in } \overline{C}, \qquad 0 < \alpha \leqq 1,$$

where M is a constant independent of n and z, and p is a non-negative integer. Then $f(z)$ is analytic in C and continuous in $\overline{C}$; the pth derivative $f^{(p)}(z)$ exists on C in the one-dimensional sense and satisfies the condition

$$(2) \qquad |f^{(p)}(z_1) - f^{(p)}(z_2)| \leqq L|z_1 - z_2|^{\alpha}|\log|z_1 - z_2||^{\beta}, \qquad z_1, z_2 \text{ on } C,$$

where $\beta=0$ if $\alpha<1$, and $\beta=1$ if $\alpha=1$, and where L is a constant independent of z_1 and z_2.

THEOREM II. *Let E, with boundary C, be a closed limited point set in the z-plane whose complement K is connected, and is regular in the sense that there exists a function $w=\phi(z)$ which maps K conformally but not necessarily uniformly onto $|w|>1$ so that the points at infinity in the two planes correspond to each other. Let the locus C_R: $|\phi(z)|=R>1$, consist of a finite number of mutually exterior analytic Jordan curves. Let $f(z)$ be defined in E, and for each n, $n=1, 2, \cdots$, let a polynomial $P_n(z)$ of degree n in z exist such that*

$$(3) \qquad |f(z) - P_n(z)| \leqq \frac{M}{n^{p+\alpha+1}R^n}, \qquad z \text{ in } E, \qquad 0 < \alpha \leqq 1,$$

* Presented to the Society, March 27, 1937.

where M is a constant independent of n and z, and p is a non-negative integer. Then $f(z)$ when suitably defined exterior to E is analytic in C_R and continuous in $\overline{C}_R$; the pth derivative $f^{(p)}(z)$ exists on C_R in the one dimensional sense and satisfies the condition

$$|f^{(p)}(z_1) - f^{(p)}(z_2)| \leqq L|z_1 - z_2|^{\alpha}|\log|z_1 - z_2||^{\beta}, \quad z_1, z_2 \text{ on } C_R, \tag{4}$$

where $\beta=0$ if $\alpha<1$, and $\beta=1$ if $\alpha=1$, and L is a constant independent of z_1 and z_2.

In these theorems the case $p=0$ is not excluded:

$$f^{(0)}(z) \equiv f(z).$$

Sewell* has already proved the slightly less general results that under the hypotheses of Theorems I and II the function $f^{(p)}(z)$ satisfies a Lipschitz condition† of every order $\alpha'<\alpha$ on C and C_R, respectively.

J. Curtiss has shown‡ that if the boundary C of a closed limited point set E consists of a finite number of mutually exterior analytic Jordan curves, if $f(z)$ is analytic in the interior points of E and continuous in E, and if $f^{(p)}(z)$ exists in the one-dimensional sense on C and satisfies on C a Lipschitz condition of order α, $0<\alpha\leqq 1$, then there exist polynomials $P_n(z)$ of respective degrees n such that (1) is valid for z in E with $\beta=0$ even for $\alpha=1$. Thus *Theorem* I *is for $0<\alpha<1$ an exact converse of Curtiss's result.* We show by an example that for $\alpha=1$ an exact converse is impossible.

To be sure, Curtiss did not state his result in the above form, *but assumed $f^{(p)}(z)$ continuous in E.* Nevertheless the lighter assumption is sufficient for his purposes, because we prove below (Theorem III) that this lighter assumption implies also the heavier assumption.

In the notation of Theorem II suppose $|f(z)-P_n(z)| \leqq \epsilon_n$, z on E, $n=1, 2, \cdots$, where ϵ_n approaches zero as n becomes infinite. The study of the relation between ϵ_n and the continuity

* Transactions of this Society, vol. 41 (1937), pp. 84–123.

† The function $f(z)$ is said to satisfy a Lipschitz condition of order α, $0<\alpha\leqq 1$, on the set E if for z_1 and z_2 on E, we have $|f(z_1)-f(z_2)| \leqq L|z_1-z_2|^{\alpha}$, where L is a constant independent of z_1 and z_2.

‡ This Bulletin, vol. 42 (1936), pp. 873–878.

properties of $f(z)$ on E is called Problem α, and the study of the relation between ϵ_n and the continuity properties of $f(z)$ on C_E is called* Problem β. Thus in Theorem I we have a result on Problem α and in Theorem II a result on Problem β.

The method of proof of Theorems I and II is an application of conformal mapping to de la Vallée Poussin's† results on trigonometric approximation.

2. *Proof of Theorem* I. The analyticity of $f(z)$ interior to C and the continuity of $f(z)$ in $\overline{C}$ follow directly from inequality (1). We consider first C to be the unit circle: $|z|=1$. Let $f(e^{i\theta}) \equiv u(\theta)+iv(\theta)$; then inequality (1) on C implies

$$(5) \qquad |u(\theta) - p_n(\theta)| \leqq \frac{M}{n^{p+\alpha}}, \qquad |v(\theta) - q_n(\theta)| \leqq \frac{M}{n^{p+\alpha}},$$

where $P_n(e^{i\theta}) \equiv p_n(\theta)+iq_n(\theta)$. But $p_n(\theta)$ and $q_n(\theta)$ are trigonometric sums of order n and hence (de la Vallée Poussin, loc. cit., pp. 57 and 61–62) we have

$$(6) \qquad \begin{aligned} |u^{(p)}(\theta_1) - u^{(p)}(\theta_2)| &\leqq L_1 |\theta_1 - \theta_2|^{\alpha} |\log|\theta_1 - \theta_2||^{\beta}, \\ |v^{(p)}(\theta_1) - v^{(p)}(\theta_2)| &\leqq L_2 |\theta_1 - \theta_2|^{\alpha} |\log|\theta_1 - \theta_2||^{\beta}, \end{aligned}$$

where $\beta=0$ if $\alpha<1$ and $\beta=1$ if $\alpha=1$. Here $u^{(p)}(\theta)$ denotes the pth derivative of $u(\theta)$ with respect to θ; $u^{(0)}(\theta) \equiv u(\theta)$. We have

$$f'(e^{i\theta}) = \frac{df(e^{i\theta})}{de^{i\theta}} = \frac{d}{d\theta}(u(\theta) + iv(\theta)) \cdot \frac{d\theta}{de^{i\theta}} = (u'(\theta) + iv'(\theta)) \frac{1}{ie^{i\theta}},$$

$$f''(e^{i\theta}) = \frac{df'(e^{i\theta})}{de^{i\theta}} = -\frac{1}{ie^{2i\theta}} [i(u''(\theta) + iv''(\theta)) + (u'(\theta) + iv'(\theta))],$$

and similarly for higher derivatives.

Now $u^{(k)}(\theta)$, $k=0, 1, 2, \cdots, p-1$, satisfies a Lipschitz condition of order 1, as does the function $e^{-ki\theta}$. Thus we have

$$|f^{(p)}(e^{i\theta_1}) - f^{(p)}(e^{i\theta_2})| \leqq L_3 |\theta_1 - \theta_2|^{\alpha} |\log|\theta_1 - \theta_2||^{\beta},$$

which through the properties of $e^{i\theta}$ implies inequality (2) for z on C; the proof of Theorem I is complete for C the unit circle.

* A more general formulation of these problems is given by Sewell, loc. cit., along with an extensive bibliography.

† Ch. J. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réele*, Paris, 1919; see especially Chap. IV.

Now let C be an arbitrary analytic Jordan curve, and let the function $w=\Phi(z)$, $z=\Psi(w)$, map $\overline{C}$ conformally onto $|w|\leqq 1$; the function $\Psi(w)$ is analytic and univalent in some closed region $|w|\leqq R_0>1$. Inequality (1) implies

$$(7) \qquad |f[\Psi(w)]-P_n[\Psi(w)]|\leqq \frac{M}{n^{p+\alpha}}, \qquad |w|\leqq 1.$$

The polynomials $P_n(z)$ are uniformly bounded on C for all n, say $|P_n(z)|\leqq M_1$; then by a well known theorem* we have (notation of Theorem II) $|P_n(z)|\leqq M_1R^n$ for z on C_R. Let Γ_{R_1} be the curve $|\Phi(z)|=R_1<R_0$, $R_1>1$, in the z-plane. Choose $R>1$ so that Γ_{R_1} lies interior to C_R. Then $|P_n[\Psi(w)]|\leqq M_1R^n$ for $|w|\leqq R_1$. We have

$$P_n[\Psi(w)]-Q_m(w)=\frac{1}{2\pi i}\int_{|t|=R_1}\frac{P_n[\Psi(t)]w^{m+1}dt}{(t-w)t^{m+1}}, \qquad |w|<R_1<R_0,$$

where $Q_m(w)$ is the sum of the first $m+1$ terms of the Taylor development of $P_n[\Psi(w)]$ about the origin. This yields

$$|P_n[\Psi(w)]-Q_m(w)|\leqq \frac{M_1R^nr^{m+1}}{(R_1-r)R_1^m}, \qquad |w|\leqq r<R_1.$$

Let us choose r, $1<r<R_1$, and choose (this method is similar to that used by Curtiss, loc. cit.) $m=qn$ where q is a positive integer such that $R(r^q/R_1^q)=r_1<1$; then we have

$$(8) \qquad |P_n[\Psi(w)]-Q_{qn}(w)|\leqq \frac{M_1r}{R_1-r}r_1^n, \qquad |w|\leqq r>1.$$

Inequalities (7) and (8) yield

$$|f[\Psi(w)]-Q_{qn}(w)|\leqq \frac{M}{n^{p+\alpha}}+\frac{M_1r}{R_1-r}r_1^n, \qquad |w|\leqq 1;$$

but we have $r_1<1$, and $(M_1r)/(R_1-r)$ is a constant independent of n, so for suitably chosen M_2 we have

$$|f[\Psi(w)]-Q_{qn}(w)|\leqq \frac{M_2}{n^{p+\alpha}}=\frac{M_2q^{p+\alpha}}{(qn)^{p+\alpha}}, \qquad |w|\leqq 1.$$

* See, for example, J. L. Walsh, *Interpolation and Approximation*, Colloquium Publications of this Society, vol. 20 (1935), pp. 77–78.

Now choose $L_N(w) \equiv 0$, $N = 1, 2, \cdots, q-1$, $L_{qN'+h}(w) \equiv Q_{qN'}(w)$, $h = 0, 1, 2, \cdots, q-1$, $N' = 1, 2, \cdots$. Thus we have

$$|f[\Psi(w)] - L_N(w)| \leqq \frac{M_3}{N^{p+\alpha}}, \quad |w| \leqq 1, N = 1, 2, \cdots,$$

since when M_3 is suitably chosen we have $M_2 q^{p+\alpha}/N_1^{p+\alpha} \leqq M_3/(N_1+h)^{p+\alpha}$, $0 \leqq h \leqq q-1$. Hence inequality (2) is satisfied by $d^p f[\Psi(w)]/dw^p$ in the one-dimensional sense on $|w| = 1$. Since $\Psi(w)$ is analytic on $|w| = 1$ it follows by the method used above* that inequality (2) is valid for z on C; the proof of Theorem I is complete.

For the function $f(w) = \sum_{k=1}^{\infty} w^k/k(k-1)$ we have ($|w| \leqq 1$)

$$\left| f(w) - \sum_{k=1}^{n} \frac{w^k}{k(k-1)} \right| \leqq \sum_{k=n+1}^{\infty} \frac{1}{k(k-1)}$$

$$= \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n}, \qquad n = 1, 2, \cdots;$$

but $f(w)$ does not satisfy a Lipschitz condition of order 1 on $|w| = 1$ since the derivative $f'(w) \equiv -\log(1-w)$ becomes infinite as w approaches 1. Thus for $\alpha = 1$ the hypothesis of Theorem I does not imply a Lipschitz condition of order α; an exact converse of Curtiss's theorem for $\alpha = 1$ is impossible.

3. *Proof of Theorem* II. Let us set

$$f(z) = P_1(z) + [P_2(z) - P_1(z)] + [P_3(z) - P_2(z)] + \cdots. \tag{9}$$

Inequality (3) implies

$$|P_{n+1}(z) - P_n(z)| \leqq \frac{2M}{n^{p+\alpha+1}R^n}, \qquad z \text{ in } E,$$

$$|P_{n+1}(z) - P_n(z)| \leqq \frac{2MR}{n^{p+\alpha+1}}, \qquad z \text{ in } \overline{C}_R.$$

Thus we have from equation (9) even exterior to E

$$|f(z) - P_n(z)| \leqq \frac{M'}{n^{p+\alpha}}, \qquad z \text{ in } \overline{C}_R.$$

* Or W. E. Sewell, this Bulletin, vol. 41 (1935), pp. 111–117; especially p. 117.

Theorem II now follows from Theorem I.

If we set for $|w| \leqq R > 1$

$$f(w) = \sum_{k=1}^{\infty} \frac{w^k}{R^k k(k-1)},$$

we have for $|w| \leqq 1$

$$\left| f(w) - \sum_{k=1}^{n} \frac{w^k}{R^k k(k-1)} \right| \leqq \sum_{k=n+1}^{\infty} \frac{1}{R^k k(k-1)}$$

$$< \frac{1}{n(n+1)} \sum_{k=n+1}^{\infty} \frac{1}{R^k} < \frac{1}{R-1} \frac{1}{n^2 R^n}.$$

Yet $f(w)$ does not satisfy a Lipschitz condition of order 1 on $|w| = R$ since, as above, the derivative $f'(w)$ becomes infinite as w approaches R. As in §2 we consequently see that for $\alpha = 1$ inequality (3) does not imply a Lipschitz condition of order α.

4. *Continuity of the Derivative in $\overline{C}$.* To establish further properties of the functions $f(z)$ of Theorems I and II, and to relate these theorems to the results of Curtiss, we prove a third theorem:

THEOREM III. *Let C be an analytic Jordan curve, let the function $f(z)$ be analytic interior to C and continuous in the corresponding closed region $\overline{C}$, and let the pth derivative $f^{(p)}(z)$ exist and be continuous on C in the one-dimensional sense. Then $f^{(p)}(z)$ defined interior to C in the usual two-dimensional sense and defined on C in the one-dimensional sense is continuous throughout $\overline{C}$.*

It is sufficient to prove the theorem for the case $p = 1$, for the proof extends automatically by induction.

Let C be the unit circle; the more general case can be transformed by conformal mapping to this special case. The original hypothesis fulfilled for an analytic curve C implies the corresponding hypothesis for the unit circle, and the conclusion proved for the case of the unit circle implies the more general conclusion stated in the theorem.

The function $f(z)$ then satisfies the conditions

$$\int_C f(z) z^k dz = 0, \qquad k = 0, 1, 2, \cdots, \tag{10}$$

by Cauchy's integral theorem. If on C we set $z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$, the Fourier development of $f(z)$ on C by (10) can be written

$$(11)\qquad f(z) \sim \sum_{n=0}^{\infty} a_n e^{in\theta} = \sum_{n=0}^{\infty} a_n z^n,$$
$$a_n = \frac{1}{2\pi}\int_C f(z)e^{-in\theta}d\theta = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz.$$

By partial integration we may write, for $k=0, 1, -1, 2, -2, \cdots$,

$$(12)\qquad \int_C f'(z)z^k dz = [z^k f(z)]_C - k\int_C f(z)z^{k-1}dz.$$

The first term on the right vanishes because $f(z)$ is single-valued and continuous on C. The Fourier development on C of $f'(z)$ is therefore by (10) and (12)

$$(13)\qquad f'(z) \sim \sum_{n=1}^{\infty} na_n e^{i(n-1)\theta} = \sum_{n=1}^{\infty} na_n z^{n-1},$$

where the a_n are given by (11).

In (11) and (13) the series represent thus far only formal developments on C. But by the continuity of $f(z)$ in $\overline{C}$, the series $\sum_{n=0}^{\infty} a_n z^n$ is precisely the Taylor development of $f(z)$ valid throughout the interior of C. By differentiation it follows that $\sum_{n=1}^{\infty} na_n z^{n-1}$ is the Taylor development of $f'(z)$, likewise valid throughout the interior of C; it follows by inspection that this series is formally identical with the Fourier development (13) on C of the derivative $f'(z)$. By the continuity of $f'(z)$ on C, the second form of the development (13) is valid uniformly on C when summed by the method of arithmetic means; the corresponding sequence converges uniformly in $\overline{C}$ to the function $f'(z)$ on C and to the function $f'(z)$ interior to C. Consequently the function $f'(z)$ is continuous throughout $\overline{C}$, and the proof is complete.

Various refinements of Theorem III exist and are to be published by the present writers on another occasion.

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ON BOUNDED CONVOLUTIONS

BY E. R. VAN KAMPEN AND AUREL WINTNER

For a type of convolutions of distributions on convex curves, there has been given recently a purely geometrical convergence criterion.† In what follows, this geometrical criterion will be freed from the convexity assumption, thus obtaining in case of bounded spectra a necessary and sufficient *convergence criterion which involves only the spectra* and not the distribution functions themselves. Only one-dimensional distributions will be considered; it will be clear from the proofs that the transition to multi-dimensional cases is obvious.

If $\beta=\beta(x)$ is a distribution function, let $S(\beta)$ denote its spectrum, and $[S(\beta)]$ the diameter of this spectrum, the diameter $[T]$ of a set T being defined as the least upper bound ($\leqq+\infty$) of the mutual distances of the pairs of points contained in T. The vectorial sum‡ of two sets T_1, T_2 will be denoted by $T_1(+)T_2$, so that $S(\beta_1{}^*\beta_2)$ is the closure of $S(\beta_1)$ $(+)$ $S(\beta_2)$; and $S(\beta_1{}^*\beta_2{}^*\cdots)$ is the closure of $S(\beta_1)(+)S(\beta_2)(+)\cdots$, whenever the infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ is convergent. An infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ will be called *bounded* if

$$\sum_{n=1}^{\infty}[S(\beta_n)]<+\infty. \tag{1}$$

It is clear that the boundedness of an infinite convolution is neither necessary nor sufficient for its convergence.

If $\beta_1{}^*\beta_2{}^*\cdots$ is convergent, its spectrum $S(\beta_1{}^*\beta_2{}^*\cdots)$ is a bounded set if and only if $\beta_1{}^*\beta_2{}^*\cdots$ is a bounded convolution in the sense (1). This is implied by the fact that *an infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ is bounded if and only if*

$$[S(\beta_1{}^*\cdots\beta_n{}^*)]<\text{const.} \qquad (n=1,2,\cdots). \tag{1'}$$

† E. R. van Kampen and A. Wintner, *Convolutions of distributions on convex curves and the Riemann zeta function*, American Journal of Mathematics, vol. 59 (1937), pp. 185–186.

‡ Cf. B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Transactions of this Society, vol. 38 (1935), p. 52 and p. 56.

The latter statement is clear in the one-dimensional case. In the multidimensional case, where the direction of a maximum diameter can rotate when n varies, the proof follows by projecting these maximum diameters on the axes of a fixed coordinate system and applying then an elementary geometrical argument. The equivalent formulation (1′) of (1) will not be used in what follows.

It will be shown that a bounded infinite convolution $\beta_1^*\beta_2^* \cdots$ is convergent (absolutely convergent) if and only if the series

$$\sum_{n=1}^{\infty} x_n \tag{2}$$

is convergent (absolutely convergent) for at least one choice of the point x_n on $S(\beta_n)$, in which case (2) is uniformly (absolutely-uniformly) convergent for all choices of x_n on $S(\beta_n)$.

Those statements of this theorem which concern absolute convergence need not be proved separately, since they are obvious consequences of the corresponding statements concerning convergence.

It is clear from (1) and from the definition of the diameter of a set that if the series (2) is convergent for a single choice of x_n on $S(\beta_n)$, then (2) is uniformly convergent for all choices of x_n on $S(\beta_n)$. It is similarly seen that (2) is convergent for all choices of x_n on $S(\beta_n)$ if and only if

$$\sum_{n=1}^{\infty} \mu_1(\beta_n) \text{ is convergent,} \tag{3}$$

where

$$\mu_m(\beta) = \int_{-\infty}^{+\infty} x^m d\beta(x). \tag{4}$$

Hence, what one has to prove is that (3) is, in case (1), necessary and sufficient for the convergence of the infinite convolution $\beta_1^*\beta_2^* \cdots$. Now it is known† that if

$$S(\beta_n) \subset K, \tag{5}$$

† B. Jessen and A. Wintner, loc. cit., p. 57.

where K is a sufficiently large bounded set independent of n, then $\beta_1^*\beta_2^* \cdots$ is convergent if and only if both series

$$(6_1) \quad \sum_{n=1}^{\infty} \mu_1(\beta_n); \qquad (6_2) \quad \sum_{n=1}^{\infty} \{\mu_2(\beta_n) - (\mu_1(\beta_n))^2\}$$

are convergent. Consequently, it is sufficient to prove that

(i) if (1) is convergent, then so is (6_2);

(ii) if $\beta_1^*\beta_2^* \cdots$ is convergent and (1) is satisfied, then there exists a bounded set K satisfying (5).

In fact, it clearly follows from (i) and (ii) respectively that (3) is sufficient and necessary for the convergence of $\beta_1^*\beta_2^* \cdots$ in case the boundedness condition (1) is satisfied.

Now, it is seen from (4), where $\mu_0(\beta) = 1$, that the series (6_2) can be written in the form

$$\sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \{x - \mu_1(\beta_n)\}^2 d\beta_n(x), \quad \text{where} \quad \mu_1(\beta_n) = \int_{-\infty}^{\infty} x d\beta_n(x).$$

This series has, in view of the definition of the diameter $[S(\beta_n)]$ of the spectrum $S(\beta_n)$ of β_n, the majorant

$$\sum_{n=1}^{\infty} [S(\beta_n)]^2,$$

a series which is convergent in virtue of (1). This proves (i). On the other hand, if $\beta_1^*\beta_2^* \cdots$ is convergent, then† $\beta_n \to \lambda$ as $n \to +\infty$, where λ denotes the "unit" distribution function, that is, the distribution function the spectrum of which consists of the single point $x = 0$. Hence, if $\beta_1^*\beta_2^* \cdots$ is convergent and (1) is satisfied, then there exists for every $\epsilon > 0$ an $N = N_\epsilon$ such that $S(\beta_n)$ is, for every $n > N$, within the ϵ-vicinity of the point $x = 0$, and so (5) holds for a sufficiently large bounded set K. This proves (ii).

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† B. Jessen and A. Wintner, loc. cit., p. 56.

ON A THEOREM OF HIGHER RECIPROCITY*

BY ALBERT WHITEMAN†

1. *Introduction.* Let $\mathfrak{D}$ denote the totality of polynomials in an indeterminate x, with coefficients in a fixed Galois field $GF(p^\tau)$ of order p^τ. Let P be a primary irreducible polynomial in $\mathfrak{D}$; then, if A is any polynomial in $\mathfrak{D}$ not divisible by P, we define $\left\{A \middle| P\right\}$ as that element in $GF(p^\tau)$ for which

$$\left\{\frac{A}{P}\right\} \equiv A^{(p^{\tau\nu}-1)/(p^\tau-1)} \pmod{P},$$

where ν is the degree of P.

We have then the following theorem of reciprocity due to H. Kuhne‡ and rediscovered by Schmidt§ and Carlitz.‖

If P and Q are primary irreducible polynomials in $\mathfrak{D}$ of degree ν and ρ respectively, then

$$\left\{\frac{P}{Q}\right\} = (-1)^{\rho\nu}\left\{\frac{Q}{P}\right\}.$$

If $M = P_1^{a_1} \cdots P_k^{a_k}$ and $(A, M) = 1$ we use the definition,

$$\left\{\frac{A}{M}\right\} = \left\{\frac{A}{P_1}\right\}^{a_1} \cdots \left\{\frac{A}{P_k}\right\}^{a_k}.$$

The purpose of this note is to give a simple new proof of the following theorem:

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* Presented to the Society, February 20, 1937.

† Harrison Scholar in Mathematics, University of Pennsylvania.

‡ H. Kuhne, *Eine Wechselbeziehung zwischen Funktionen mehrerer Unbestimmter die zu Reziprozitätsgesetzen führt*, Journal für die reine und angewandte Mathematik, vol. 124 (1901–02), pp. 121–133.

§ F. K. Schmidt, *Zur Zahlentheorie in Körpern von der Charakteristik p*, Sitzungsberichte der Physikalish-medizinischen Societät zu Erlangen, vol. 58–59 (1928), pp. 159–172.

‖ L. Carlitz, *The arithmetic of polynomials in a Galois field*, American Journal of Mathematics, vol. 54 (1932), pp. 39–50.

If M and N are primary relatively prime polynomials in $\mathfrak{D}$ of degree m and n respectively, then

$$\left\{\frac{M}{N}\right\} = (-1)^{mn}\left\{\frac{N}{M}\right\}.$$

This generalized form of Kuhne's theorem is, of course, not new. The novelty of our method consists in proving the case M, N directly (rather than P, Q) by making use of the generalized analog of Gauss's lemma* proved in §2.

2. *Generalization of the Analog of Gauss's Lemma.* We shall employ the following notation. If

$$F = a_0x^\nu + a_1x^{\nu-1} + \cdots + a_\nu, \qquad a_0 \neq 0,$$

is a polynomial in $\mathfrak{D}$, then

$$\operatorname{sgn} F = a_0, \qquad \deg F = \nu;$$

for $\operatorname{sgn} F = 1$, F is said to be *primary*. Let $\mathfrak{R}(A/B)$ denote the remainder in the division of A by B. Then the generalization in question is furnished by the following lemma.

LEMMA. *Let A and M be in $\mathfrak{D}$, M primary and relatively prime to A; then*

$$\left\{\frac{A}{M}\right\} = \prod_{\deg H<m} \operatorname{sgn} \mathfrak{R}\left(\frac{HA}{M}\right),$$

the product extending over all primary H of degree less than the degree of M.

We shall now give a proof of this lemma along lines suggested by Schering's† proof in the numerical case.

3. *Proof of the Lemma.* Following Dedekind,‡ we define $\phi(M)$ to be the number of polynomials in a reduced residue system, mod M; the number of primary polynomials prime to M

* L. Carlitz, loc. cit., p. 46.

† E. Schering, *Zur Theorie der quadratischen Reste*, Acta Mathematica, vol. 1 (1882), pp. 153–170; see also P. Bachmann, *Die Elemente der Zahlentheorie*, 1892, pp. 144–148.

‡ R. Dedekind, *Abriss einer Theorie der höheren Congruenzen in Bezug auf einer reellen Primzahl-Modulus*, Journal für die reine und angewandte Mathematik, vol. 54 (1857), pp. 1–26.

and of degree less than m is then evidently $\phi(M)/(p^r-1)$. Hence, just as in the numerical case, it is very easy to show that the number of primary polynomials H of degree less than m such that $(H, M)=D$ is $\phi(M/D)/(p^r-1)$.

Put $H=H_1D$, $M=M_1D$. Then the congruence

$$HA \equiv H' \operatorname{sgn} \Re\left(\frac{HA}{M}\right) \pmod{M}, \quad \deg H' < m, \quad \operatorname{sgn} H' = 1,$$

becomes

$$H_1A \equiv H_1' \operatorname{sgn} \Re\left(\frac{HA}{M}\right) \pmod{M_1}. \tag{1}$$

Evidently the polynomials H_1 are the polynomials H_1' in some order. Therefore, if we multiply all congruences of the type (1) together and divide each member of the resulting congruence by the product of the H_1 (which is prime to M_1), we have

$$A^{\phi(M_1)/(p^r-1)} \equiv \prod_{(H,M)=D} \operatorname{sgn} \Re\left(\frac{HA}{M}\right) \pmod{M_1}. \tag{2}$$

For $M_1=P$, a primary irreducible polynomial of degree ν, the last congruence becomes

$$A^{(p^{r\nu}-1)/(p^r-1)} \equiv \left\{\frac{A}{P}\right\} \pmod{P}. \tag{3}$$

Writing this congruence in the form

$$A^{(p^{r\nu}-1)/(p^r-1)} = \left\{\frac{A}{P}\right\} + FP,$$

and raising both members to the $p^{r(k-1)\nu}$th power, we can readily show that

$$A^{p^{r(k-1)\nu}(p^{r\nu}-1)/(p^r-1)} = \left\{\frac{A}{P}\right\} + F'P^{p^{r(k-1)\nu}}.$$

But it is well known that

$$\phi(P^k) = p^{r(k-1)\nu}(p^{r\nu} - 1).$$

Hence

$$A^{\phi(P^k)/(p^\tau-1)} \equiv \left\{\frac{A}{P}\right\} \pmod{P^k}. \tag{4}$$

Finally, for $M_1 = P_1^{b_1} \cdots P_k^{b_k}$, $0 \leqq b_i \leqq a_i$, $k > 1$, deg $P_i = \nu_i$, we have

$$\frac{\phi(M_1)}{p^\tau - 1} = \frac{1}{p^\tau - 1} \prod_{i=1}^{k} p^{(b_i-1)\tau\nu_i}(p^{\tau\nu_i} - 1).$$

Hence, since

$$A^{p^{\tau\nu_i}} \equiv 1 \pmod{P_i},$$

it follows that

$$A^{\phi(M_1)/(p^\tau-1)} \equiv 1 \pmod{M_1}, \tag{5}$$

where, as already stated, M_1 is the product of at least two distinct irreducible polynomials.

Combining the results of (2), $\cdots$, (5) we now see that

$$\coprod_{(H,M)=D} \operatorname{sgn} \mathfrak{R}\left(\frac{HA}{M}\right) \tag{6}$$

has the value 1 unless $M_1 = M/D$ is irreducible or the power of an irreducible polynomial. On the other hand, for $M_1 = P_i^b$ $(b = 1, \cdots, a_i)$, (6) has the value $\left\{A \,|\, P_i\right\}$. Consequently

$$\coprod_{D|M} \coprod_{(H,M)=D} \operatorname{sgn} \mathfrak{R}\left(\frac{HA}{M}\right) = \coprod_{\deg H < m} \operatorname{sgn} \mathfrak{R}\left(\frac{HA}{M}\right)$$
$$= \left\{\frac{A}{P_1}\right\}^{a_1} \cdots \left\{\frac{A}{P_k}\right\}^{a_k},$$

from which the Lemma follows at once.

4. *Proof of the Theorem.* Let A, N denote primary polynomials of degrees a, n respectively; let $(A, N) = 1$, $a \geqq n$. Consider the congruence

$$A \equiv \mathfrak{R}(A/N) \pmod{N}, \quad \deg \mathfrak{R}(A/N) < n.$$

Evidently there exists a primary H(say H_0) of degree $a - n$ such that

$$A = \mathfrak{R}(A/N) + H_0 N.$$

But this equation may be written in the form

$$H_0N \equiv -\mathcal{R}(A/N) \pmod{A}. \tag{7}$$

Let E be any polynomial (not necessarily primary) of degree less than $a-n$. Then we may write

$$(H_0 + E)N \equiv EN - \mathcal{R}(A/N) \pmod{A}, \tag{8}$$

where

$$0 < \deg(EN - \mathcal{R}(A/N)) < a, \tag{9}$$

$$\operatorname{sgn}(EN - \mathcal{R}(A/N)) = \operatorname{sgn} EN = \operatorname{sgn} E.$$

Furthermore, we have the obvious identity

$$\prod_{\deg H=a-n} HN = H_0N \prod_{\deg E<a-n} (H_0 + E)N, \qquad E \neq 0. \tag{10}$$

Therefore, by equations (7), $\cdots$, (10),

$$\begin{aligned} &\prod_{\deg H=a-n} \operatorname{sgn} \mathcal{R}\left(\frac{HN}{A}\right) \\ &= \operatorname{sgn} \mathcal{R}\left(\frac{H_0N}{A}\right) \prod_{\deg E<a-n} \operatorname{sgn} \mathcal{R}\left(\frac{(H_0 + E)N}{A}\right) \\ &= -\operatorname{sgn} \mathcal{R}\left(\frac{A}{N}\right) \prod_{\deg E<a-n} \operatorname{sgn} E. \end{aligned} \tag{11}$$

Now, by the generalization of Wilson's theorem for a Galois field,

$$\prod_b b = -1, \qquad b \text{ in } GF(p^r),$$

from which it follows at once that

$$\prod_{\deg E<a-n} \operatorname{sgn} E = (-1)^{a-n}.$$

Hence (11) becomes

$$\prod_{\deg H=a-n} \operatorname{sgn} \mathcal{R}\left(\frac{HN}{A}\right) = (-1)^{a-n+1} \operatorname{sgn} \mathcal{R}\left(\frac{A}{N}\right). \tag{12}$$

Since

$$\mathcal{R}\left(\frac{HN}{A}\right) = -\mathcal{R}\left(\frac{A}{HN}\right), \deg HN = \deg A,$$

(12) may also be written in the form

$$(13)\qquad \prod_{\deg H=a-n} \operatorname{sgn} \mathfrak{R}\left(\frac{A}{HN}\right) = (-1)^{a-n} \operatorname{sgn} \mathfrak{R}\left(\frac{A}{N}\right).$$

Let us now assume, as we may without any loss of generality, that $m \geqq n$. In (12) replace A by KM, where K is any primary polynomial of degree $k (k < n)$. Then we have

$$\prod_{\deg H=k+m-n} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{KM}\right) = (-1)^{k+m-n+1} \operatorname{sgn} \mathfrak{R}\left(\frac{KM}{N}\right).$$

Now let K run through all the p^{rk} primary polynomials of degree k; we get

$$(14)\qquad \prod_{\substack{\deg H=k+m-n \\ \deg K=k}} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{KM}\right) = (-1)^{k+m-n+1} \prod_{\deg K=k} \operatorname{sgn} \mathfrak{R}\left(\frac{KM}{N}\right).$$

In a similar manner we may obtain from (13),

$$(15)\qquad \prod_{\substack{\deg H=k+m-n \\ \deg K=k}} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{KM}\right) = (-1)^{k} \prod_{\deg H=k+m-n} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{M}\right).$$

Comparing (14) and (15), we obtain

$$\prod_{\deg K=k} \operatorname{sgn} \mathfrak{R}\left(\frac{KM}{N}\right) = (-1)^{m+n-1} \prod_{\deg H=k+m-n} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{M}\right).$$

Therefore

$$\prod_{\deg K<n} \operatorname{sgn} \mathfrak{R}\left(\frac{KM}{N}\right) = (-1)^{mn+n^2-n} \prod_{m-n \leqq \deg H<m} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{M}\right).$$

When we note that

$$\prod_{\deg H<m-n} \operatorname{sgn} \mathfrak{R}\left(\frac{HN}{M}\right) = 1,$$

the theorem follows at once.

UNIVERSITY OF PENNSYLVANIA

THE MOST GENERAL TRANSFORMATIONS OF PLANE REGIONS WHICH TRANSFORM CIRCLES INTO CIRCLES

BY CONSTANTIN CARATHÉODORY

1. *Introduction.* If we consider two plane regions R and R' which are mapped conformally the one upon the other, there corresponds to every circle c contained in R an analytic closed curve c' contained in R'. If an arc α lying on the circle c has a circular image the curve c' must be itself a circle.

Suppose that the interior of the circle c belongs to the region R. Then both circular discs bounded by c and c' respectively are represented conformally the one upon the other. It is a well known fact that in this case the transformation of these circles into one another is given by a transformation of Möbius by which, furthermore, every circle of R is transformed into a circle of R'.

If we drop now the condition that the one to one correspondence of our regions must be conformal, the assumption that one single circle of R has a circular image is no longer sufficient to characterize the transformations of Möbius. On the other hand, if we make the stronger assumption that *every* closed circle contained in R is transformed into a circle, we shall see in the course of this paper that a theorem analogous to the one stated above holds under surprisingly general conditions. To prove that the transformation which we consider is a transformation of Möbius it is no longer necessary to assume from the outset (as is the case for the analogous theorem concerning collinear transformations) that this transformation is continuous, or that it is measurable in the sense of Lebesgue, or even that the point set R' is itself a region.

The condition that circles lying in a region R have circular images characterizes the group of Möbius transformations among all the one to one arbitrary correspondences between the points of R and the points of a quite arbitrary point set of the plane.

2. *Statement of a Preliminary Theorem.* We consider a circular open disc which we shall call C and suppose that there is a one

to one correspondence between the points P of this disc and the points P' of a *bounded* point set C'. We do not suppose this correspondence to be continuous or even to be measurable in the sense of Lebesgue. But we assume that every closed circular line c which is contained in the domain C is transformed by the above correspondence into a line c' of the same kind whose points lie on C'. We shall prove that every transformation which possesses these properties is a Möbius transformation and therefore analytic.

It is very important to make the following rather obvious remark: if two circles c_1' and c_2' contained in C' are the images of two circles of C, say of c_1 and c_2, then the number of points common to the pair of circles c_1' and c_2' is the same as the number of points common to c_1 and c_2. In particular, if c_1 and c_2 are tangent to one another the same is the case for c_1' and c_2' and conversely.

3. *Reduction of the Transformation to a Normal Type.* In using transformations of M bius we plot the domain C and a circle D' which contains in its interior the bounded point set C' on the unit circles of two planes with coordinates x, y and x', y' respectively. In doing this we can always assume that the origins of both planes correspond to points of C and of C' which are images of one another by the given transformation.

Finally we make two inversions respectively on the unit circles of both these planes.

4. *Preservation of Parallelism.* To the original transformation of the domain C into the point set C' corresponds now a one to one transformation of the exterior E of the unit circle in the plane with coordinates x, y into some point set E' lying outside of the unit circle of the second plane. To every closed circular line lying in E corresponds a closed circle of E'. Finally every straight line contained entirely in E is transformed into a straight line lying in E'.

Likewise every pair of parallel straight lines of E is transformed into a pair of parallel straight lines of E'.

5. *Preservation of Orthogonality.* Take now two such pairs of parallel straight lines lying in E and cutting at right angles. Suppose furthermore that there is a circle contained entirely in E and circumscribed to the rectangle formed by these lines.

The whole figure is transformed into a circle of E' in which a parallelogram is inscribed; this parallelogram must therefore be a rectangle.

We infer herefrom without difficulty that two straight lines of E orthogonal to one another are always transformed into lines of E' having the same property.

6. *The Image of C is a Domain.* We take two circles of E which are concentric to the unit circle and consider two or more rectangles inscribed in the one of these circles and having two opposite sides tangent to the other. As the corresponding figure lying on E' shows exactly the same disposition, the circles contained in these figures and which are images of the two circles considered above must also be concentric to one another.

Furthermore let c be an arbitrary circle of E concentric to the unit circle. To every point P' lying on the image c' of c corresponds a point P of c. To the tangent to the circle c passing through the point P corresponds the tangent to c' passing through P'. Consequently every point of the plane with coordinates x', y' exterior to c' must belong to the point set E'. We conclude from this that E' is an open set consisting of all the points which are exterior to a certain circle. The point set C' which we have considered above must therefore be a circular disc and we could have taken the circle D' coinciding with the boundary of C'.

If we do this the point set E' coincides with the exterior of the unit circle of our plane with coordinates x', y', and the circles of E concentric to the unit circle are transformed into circles of E' having the same property.

7. *Preservation of the Centers of Circles.* Take now a circle γ lying in E whose center is at a distance from the origin greater than $2^{1/2}$. Under these conditions there are at least two diameters of γ orthogonal to one another which belong to straight lines lying in E. There are also at least two tangents to the same circle cutting at right angles and parallel to the diameters which we have been considering and entirely contained in E.

This figure transforms into a figure contained in E' from whose inspection we conclude that if we call γ' the image of γ, the center of γ' is the image of the center of γ.

8. *Preservation of Angles at the Origin.* We take now the positive number

$$r > 2^{1/2}, \tag{1}$$

and consider the circle

$$x^2 + y^2 = r^2 \tag{2}$$

and the circle

$$x'^2 + y'^2 = r'^2 \tag{3}$$

into which the circle (2) is transformed. We will show that every regular polygon inscribed in (2) and whose sides have a length less than $(r-1)$ is transformed into a regular polygon inscribed in (3) and having the same number of vertices.

Two adjoining sides of the former polygon must in fact, by the result of the last paragraph, be transformed into contiguous segments of equal length inscribed in (3). The second polygon is therefore either a regular polygon similar to the first or it is starshaped. But in this latter case at least two sides of it would cross at a point interior to the circle (3) and such a point must necessarily be the image of a point exterior to the circle (2). But this is readily shown to be impossible by using a reasoning like that of §6. Thus it is proved that the two polygons are similar to one another.

We apply this to all such polygons inscribed in (2) which have a common vertex at the point $x=r$, $y=0$.

Any point of the circle (2) which satisfies the condition

$$x + iy = r \cdot e^{2\pi ip/q} \tag{4}$$

where p and q are arbitrary integers and $q \neq 0$, can be considered as a vertex of one at least of these polygons. And we can always choose the coordinates x' and y' in such a way that the images of all the points (4) must be calculated by the formulas

$$\frac{x'}{r'} = \frac{x}{r}, \qquad \frac{y'}{r'} = \frac{y}{r}. \tag{5}$$

9. *Families of Circles which Are Transformed by a Similarity.* Take now all the tangents to the circle (2) whose points of contact coincide with the point set (4) and call A the set of points of intersection of each two of these tangents. Each point of the

set A will then be transformed into a certain point contained in the exterior of the circle (3) whose coordinates must be calculated by (5).

Call A^* the family of circles, each of which is contained in E and passes through three points at least of the set A. The image of every one of these circles is necessarily connected with the circle itself by the transformation (5).

We remark finally that the point set A is everywhere dense in the exterior of the circle (2) and that therefore every circle lying in E and possessing points outside of (2) can be indefinitely approximated by circles belonging to the family A^*.

10. *Proof of the Continuity of the Transformation.* Our next step is to show that our transformation is necessarily continuous along the circle (2). If this were not the case there would exist on this circle at least one point P_0 whose image on the circle (3) we obtain by a rotation around the origin through an angle θ lying between zero and 2π. We could then construct a point set B analogous to our former point set A and a set of circles B^* analogous to A^* and possessing the following property: To obtain the image of a circle belonging to B^* we must combine the amplification of §9 according to the cases either with a rotation around the origin through the angle θ or with a reflection on some diameter of the circle (3).

In both these cases it is always possible to find two circles, say c_1 and c_2, both lying in the region

$$x^2 + y^2 > r^2, \tag{6}$$

cutting one another, and such that if we transform c_1 by the amplification of §9 and c_2 by one of the transformations which have just been described, we obtain as result of these transformations circles which have no point in common.

We now approximate c_1 by a circle c_1^* belonging to A^* and c_2 by a circle c_2^* belonging to B^*. If these approximations are close enough the circles c_1^* and c_2^* will cut one another and will be transformed by the transformation we study into circles having no common point. But this is in contradiction with the general principle laid down at the end of §2.

We must therefore assume that all the points of the circle (2) are transformed into the points of (3) by the transformation

(5) and the same is then the case for all the points of the region (6).

11. *The Theorem for the Circle.* Take now a point P of E lying between the unit circle and the circle (2). Consider two circles of E passing through P and possessing points inside of the region (6). By hypothesis these circles are transformed into circles and besides they contain arcs whose images we know.

The result is that the formulas (5) must hold for every point of the region E. As this region must by assumption be transformed into E' we find that we must have

$$r' = r. \tag{7}$$

In short we have proved the following theorem:

THEOREM 1. *Every arbitrary one to one correspondence between the points of a circular disc C and a bounded point set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a direct or an inverse transformation of Möbius.*

12. *The General Theorem.* We now consider a general region R and suppose that there exists a one to one correspondence between the points of R and some point set R'. We assume furthermore that by this correspondence circles of R are transformed into circles lying on R'. We call c and c' two such corresponding circles and suppose that every point interior to c belongs to R. Through every pair of points P and Q lying in the interior of c there passes at least one circle c_1 having no points in common with c. To this circle corresponds a circle c_1' which has no point in common with c'. Therefore both points P' and Q' which correspond to P and Q must lie on the same side of c'. It follows that the interior of c must correspond to a point set which lies on one side of c'. This point set is either itself bounded, or it can be transformed by an inversion on the circle c' into a bounded set. In both cases we can apply the previous theorem. As the circle c can be taken at random we have finally the following theorem.

THEOREM 2. *Suppose that to every point P of a region R corresponds a point P' of some point set R' and that to two different points P and Q of R correspond two points P' and Q' of R' differ-*

ent from one another. Suppose that to every circle c contained with its interior in the region R corresponds a point set of R′ which consists of all the points of a closed circle c′. Then the point set R′ is itself a region and the transformation of R into R′ is analytic and either a direct or an inverse transformation of Möbius.

It is not difficult to generalize this result by restricting the class of circles c belonging to R which are supposed to be transformed into circles. Take for instance a continuous positive function $\phi(P)$ defined everywhere in the region R. Then the theorem holds if we suppose that every circle of center P and whose radius is less than $\phi(P)$ is transformed into a circle lying on R'.

The following generalization of Theorem 2 is nearly self evident if we note that three circles in space cutting one another at six different points must lie on the same sphere.

THEOREM 3. *Supposing that a plane region R is transformed by a one to one correspondence into an arbitrary point set R′ lying in n-space* ($n \geqq 3$) *under the same assumptions as before, then R′ must be a two dimensional sphere or a plane and the transformation is a transformation of Möbius.*

Other generalizations can also be imagined which, however, are outside the scope of this note.

UNIVERSITY OF MUNICH

A NOTE ON DETERMINANTS

BY M. H. INGRAHAM

The following theorem, with its corollary, when applicable affords great numerical simplifications. Although the writer has tried and failed to discover it in the literature, it is submitted with some hesitation lest it may not be new. The corollary was discovered and proved by the author. Upon seeing it Professor Max Zorn suggested the more general form of the theorem, which was then proved by the author.

THEOREM. *If A is a square matrix* (A_{ij}), $(i, j = 1, \cdots, n)$, *where the* A_{ij} *are* $m \times m$ *matrices which are commutative in pairs, and if B is the* $m \times m$ *matrix which is arrived at by taking the*

determinant of A with the A_{ij} as elements, then the determinant of B is equal to the determinant of A.

COROLLARY. *If $(G(x)=g_{ij}(x))$ is a square matrix whose elements are polynomials $g_{ij}(x)$ and if M is a square matrix of constant elements, then if the determinant of $G(x)$ is $g(x)$, the determinant of $G(M)$ (that is, the matrix of matrices $g_{ij}(M)$) is the determinant of $g(M)$.*

The method of proof is by induction on n. We will first consider the case where the field to which the elements of the A_{ij} belong has an infinite number of elements. It will be sufficient to show that the theorem holds when A is replaced by $A-\lambda I$ and A_{ii} by $A_{ii}-\lambda I$, where $A_{11}-\lambda I$ is non-singular, since the identity established will hold for an infinitude of values of λ and hence for $\lambda=0$.

Consider $A^{(1)}=(A-\lambda I)C$ where

$$C=\begin{pmatrix} I, & -(A_{11}-\lambda I)^{-1}A_{12}, & -(A_{11}-\lambda I)^{-1}A_{13}, \cdots & -(A_{11}-\lambda I)^{-1}A_{1n} \\ 0 & I & 0 \cdots & 0 \\ 0 & 0 & I \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots & I \end{pmatrix}.$$

We see that $A^{(1)}$ is a matrix of the same type as A, and $|A^{(1)}|=|A-\lambda I|$, since $|C|=1$. If $B^{(1)}$ and $B^{(2)}$ are derived from $A^{(1)}$ and $A-\lambda I$ respectively in the same way that B is from A, then $B^{(1)}=B^{(2)}$. But $A^{(1)}$ is such that $A_{11}^{(1)}=A_{11}-\lambda I$ and $A_{11}^{(1)}=0$, $(i\neq 1)$, so that $B^{(1)}=(A_{11}^{(1)}-\lambda I)D$ where D is derived from the minor matrix of $A_{11}^{(1)}-\lambda I$ in the same way as B from A, and $|A^{(1)}|=|A_{11}^{(1)}|\,|D|$ since our proof by induction assumes that the theorem holds for $(n-1)\times(n-1)$ matrices of matrices. But $|B^{(2)}|=|B^{(1)}|=|A_{11}|\,|D|=|A^{(1)}|=|A-\lambda I|$. Hence the theorem follows for infinite fields.

As all finite fields are Galois fields derived from the field of all rational functions, an identity of above type that holds in the latter field is also true for Galois fields.

UNIVERSITY OF WISCONSIN

A NEW DEFINITION OF A STIELTJES INTEGRAL*

BY A. H. COPELAND

Let $f(x)$ be a function of limited variation defined on (α, β), and $x_1, x_2, \cdots, x_n$ a sequence which divides (α, β) into a finite number of intervals. Let $g(x)$ be bounded on (α, β). If

$$\text{(A)} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\{f(x_i) - f(x_{i-1})\}$$

exists when $x_i - x_{i-1}$ tends to zero, this limit is the Riemann-Stieltjes integral of g with respect to f. If g is continuous the integral exists. If g is not continuous (A) need not exist. In particular (A) does not exist when g is of bounded variation unless restrictions which are conditioned by g are placed on the sequence $x_1, x_2, \cdots$.

In this note we define an integral of g with respect to f as the Cesàro mean of g formed for a sequence $x_1, x_2, \cdots$ of ordinates. This sequence is defined in terms of f only. When the function f, and consequently the method of formation of the sequence $x_1, x_2, \cdots$, has been given, the integral exists for every g which has discontinuities of the first kind only, and reduces to (A) when g is continuous. For the present we shall assume that f is monotone, $f(\alpha+0)=0$, and $f(\beta)=1$. At a later point we shall show how these restrictions can be removed.

The sequence $x_1, x_2, \cdots$ is related to the function f in the following manner: Let $a<x\leqq b$ be any sub-interval of $\alpha<x\leqq\beta$. Let H_n be the number of points of the sequence $x_1, x_2, \cdots, x_n$

* Presented to the Society, December 31, 1930. For other extensions of the concept of the Stieltjes integral, see H. L. Smith, *On the existence of the Stieltjes integral*, Transactions of this Society, vol. 27 (1925), p. 491; S. Pollard, *The Stieltjes integral and its generalizations*, Quarterly Journal of Mathematics, vol. 49 (1920–3), p. 73; Kolmogoroff, *Untersuchungen über den Integralbegriff*, Mathematische Annalen, vol. 103 (1930), p. 654; Young, *The algebra of many-valued quantities*, Mathematische Annalen, vol. 104 (1931), p. 260, and *Many-valued Riemann-Stieltjes integration*, Cambridge Philosophical Society Proceedings, vol. 27 (1931), p. 325; Ben Dushnik, *On the Stieltjes integral*, published by Edwards Brothers (Ann Arbor).

which fall in (a, b). Then H_n/n is the relative frequency with which the terms of the sequence fall in (a, b), and

$$\lim_{n\to\infty} \frac{H_n}{n} = f(b+0) - f(a+0). \tag{1}$$

A similar condition holds for closed intervals, open intervals, and intervals closed on the left and open on the right.

Let $G_n = g(x_1) + g(x_2) + \cdots + g(x_n)$. If the sequence $x_1, x_2, \cdots$ satisfies (1) the integral is defined by*

$$\lim_{n\to\infty} \frac{G_n}{n} = \int_a^b g(x)df(x), \tag{2}$$

provided this limit exists.

In order to construct the sequence† $x_1, x_2, \cdots, x_n$ we shall group the terms as follows: x_1; x_2, x_3; x_4, x_5, x_6; $\cdots$. The sth group contains the terms x_i whose subscripts have the form $i = s(s-1)/2 + k$ where $k = 1, 2, 3, \cdots, s$. Let x_i be the greatest lower bound of the numbers x for which

$$f(x-0) \leqq (2k-1)/2s \leqq f(x+0).$$

Then $f(x_i - 0) \leqq (2k-1)/2s \leqq f(x_i + 0)$ and $a < x_i \leqq b$ if and only if $f(a+0) < (2k-1)/2s \leqq f(b+0)$. If z_s is the number of terms of the sth group lying in (a, b), then z_s is the number of integers k for which these last inequalities hold. Simple arithmetic then gives

$$[f(b+0) - f(a+0)]s - 1 < z_s < [f(b+0) - f(a+0)]s + 1.$$

Therefore

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* $\lim_{n\to\infty} G_n/n$ is the arithmetic or first Cesàro mean of the ordinates $g(x_1)$, $g(x_2)$, $\cdots$. In this connection we note that J. C. Burkill has developed a definition of an integral based on Cesàro means and Perron's integral. See Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 314–322; vol. 39 (1935), pp. 541–552; Journal of the London Mathematical Society, vol. 10 (1935), pp. 255–259; vol. 11 (1936), pp. 43–88, 220–226.

† The method which we shall describe is similar to that used by von Mises in the second example of his article *Über Zahlenfolgen, die ein kollektiv-ähnliches Verhalten zeigen*, Mathematische Annalen, vol. 108, no. 5 (1933). An alternative construction can be found in the author's article *A matrix theory of measurement*, Mathematische Zeitschrift, vol. 37, no. 4 (1933).

$$[f(b+0)-f(a+0)]s(s+1)/2-s<h_s$$
$$<[f(b+0)-f(a+0)]s(s+1)/2+s,$$

where $h_s=z_1+z_2+\cdots+z_s$. If $s(s+1)/2\geqq n>s(s-1)/2$, then $h_{s-1}\leqq H_n\leqq h_s$ and hence

$$[f(b+0)-f(a+0)]s(s-1)/2-(s-1)<H_n$$
$$<[f(b+0)-f(a+0)]s(s+1)/2+s.$$

If we divide the left-hand member of this inequality by $s(s+1)/2$, the middle member by n, and the right-hand member by $s(s-1)/2$, we obtain

$$[f(b+0)-f(a+0)]\frac{s-1}{s+1}-\frac{2(s-1)}{s(s+1)}<\frac{H_n}{n}$$
$$<[f(b+0)-f(a+0)]\frac{s+1}{s-1}+\frac{2}{s-1}.$$

Therefore

$$\lim_{n\to\infty} H_n/n=f(b+0)-f(a+0).$$

Similarly, since the number of integers k satisfying the inequalities $f(a-0)\leqq(2k-1)/2s\leqq f(a+0)$ is greater than $[f(a+0)-f(a-0)]s-1$ and less than $[f(a+0)-f(a-0)]s+1$, it follows that the relative frequency with which a number a is repeated in the sequence $x_1, x_2, \cdots$ tends to $f(a+0)-f(a-0)$ as n becomes infinite. Thus conditions (1) are satisfied with respect to closed, open, and half open intervals.

We shall next prove the existence of the integral. Let $a<x\leqq b$ be any sub-interval of (α, β) and let G_n' be the sum of those terms of the set $g(x_1), g(x_2), \cdots, g(x_n)$ for which the corresponding ordinates lie in (a, b). Then $mH_n\leqq G_n'\leqq MH_n$ where M and m are the maximum and minimum* of $g(x)$ in (a, b). Thus the superior and inferior limits of G_n'/n both lie between $m[f(b+0)-f(a+0)]$ and $M[f(b+0)-f(a+0)]$. Hence these limits differ by at most $(M-m)[f(b+0)-f(a+0)]$. If G_n'' is the sum of those terms of $g(x_1), g(x_2), \cdots, g(x_n)$ for which the corresponding points of $x_1, x_2, \cdots$ are equal to some number c, then G_n''/n approaches the unique limit $g(c)[f(c+0)-f(c-0)]$.

* In case $g(x)$ does not possess a maximum and minimum in (a, b), then M and m shall designate the least upper and greatest lower bounds of g in this interval.

If ϵ is a given positive number and c is any point of (α, β) for which the saltus of $g(x)$ is less than ϵ, then c can be enclosed in an interval (a, b) within which the oscillation of g is less than ϵ. It follows that the superior and inferior limits of G_n'/n differ by at most $\epsilon[f(b+0)-f(a+0)]$. If the saltus of g at a point c is greater than or equal to ϵ, then, since the discontinuity of g at c is of the first kind, we can choose two points a and b for which the oscillation of g is less than $\epsilon/2$ in each of the intervals $a<x<c$ and $c<x\leqq b$. Hence the superior and inferior limits of G_n'/n for the intervals (a, c) and (c, b) differ respectively by at most $[f(c-0)-f(a-0)]\epsilon/2$ and $[f(b+0)-f(c+0)]\epsilon/2$. Since the superior and inferior limits of G_n''/n for the point c differ by 0, it follows that the superior and inferior limits of G_n'/n for the interval (a, b) differ by at most $\epsilon[f(b+0)-f(a+0)]$.

Since $g(x)$ has discontinuities of the first kind only, it follows that the points at which the saltus of g is greater than ϵ are finite in number. Consequently the interval (α, β) can be divided into a finite sum of intervals such that the superior and inferior limits of G_n'/n in the ith interval (a_{i-1}, a_i) of the decomposition differ by at most $\epsilon[f(a_i+0)-f(a_{i-1}+0)]$. It follows that the superior and inferior limits of G_n/n differ by at most $\epsilon\sum[f(a_i+0)-f(a_{i-1}+0)]=\epsilon[f(\beta)-f(\alpha+0)]=\epsilon$. Thus the limit of G_n/n exists. Moreover if $x_1, x_2, \cdots$ and $\bar{x}_1, \bar{x}_2, \cdots$ are two sequences satisfying conditions (1) with respect to $f(x)$, and if G_n' and $\overline{G}_n'$ are corresponding sums for the interval (a_{i-1}, a_i), then G_n'/n and $\overline{G}_n'/n$ approach limits which differ by at most $\epsilon[f(a_i+0)-f(a_{i-1}+0)]$. It follows that G_n/n and $\overline{G}_n/n$ approach the same limits.

If $f(x)$ is monotone but fails to satisfy the conditions $f(\alpha+0)=0$ and $f(\beta)=1$, then we can define the integral by means of the equation

$$(3) \qquad \int_\alpha^\beta g(x)df(x) = [f(\beta) - f(\alpha+0)]\int_\alpha^\beta g(x)d\phi(x),$$

where $\phi(x)=[f(x)-f(\alpha+0)]/[f(\beta)-f(\alpha+0)]$. If $f(x)$ is an arbitrary function of limited variation, we can define the integral as the difference of the integrals with respect to the positive and negative variations of f. Thus if $g(x)$ has discontinuities of the first kind only and if $f(x)$ is of limited variation, then the integral exists.

In order to form the integral over a sub-interval (α', β') of (α, β), the following device is convenient. Let $\gamma(x)$ be a function which is equal to $g(x)$ when x is in (α', β') and equal to 0 otherwise. Then the integral of $\gamma(x)$ extended over the interval (α, β) is equal to the integral of $g(x)$ extended over the interval (α', β'). It is however necessary to investigate whether or not this definition is consistent with (2). We have

$$\int_\alpha^\beta \gamma df = \{f(\beta) - f(\alpha + 0)\} \int_\alpha^\beta \gamma d\phi,$$

$$\int_{\alpha'}^{\beta'} g df = \{f(\beta') - f(\alpha' + 0)\} \int_{\alpha'}^{\beta'} g d\bar{\phi},$$

where

$$\phi = \{f(x) - f(\alpha + 0)\} / \{f(\beta) - f(\alpha + 0)\},$$

$$\bar{\phi} = \{f(x) - f(\alpha' + 0)\} / \{f(\beta')\} - f(\alpha' + 0)\}.$$

Let G_n and $\bar{G}_n$ be the sums corresponding to the first and second of these formulas respectively. Then the part of G_n which arises from the exterior of $\alpha' < x \leqq \beta'$ is zero. If (a, b) is an interval with $\alpha' < a \leqq b \leqq \beta'$, and G_n' and $\bar{G}_n'$ are the parts of G_n and $\bar{G}_n$ arising from $a < x \leqq b$, then

$$m\{f(b + 0) - f(a + 0)\} \leqq \lim_{n \to \infty} \frac{G_n'}{n} \leqq M\{f(b + 0) - f(a + 0)\},$$

and the same relations hold for $\bar{G}_n'$. The numbers m and M are the greatest lower bound and least upper bound of g on $a < x \leqq b$. The relation

$$\int_\alpha^\beta \gamma df = \int_{\alpha'}^{\beta'} g df \tag{4}$$

now follows as in the proof of the existence of $\int_\alpha^\beta g df$.

If G_n^1, G_n^2, and G_n are the sums corresponding to the functions $g_1(x)$, $g_2(x)$, and $g(x) = g_1(x) + g_2(x)$, then $G_n^1 + G_n^2 = G_n$. Hence, dividing by n and passing to the limit, we obtain

$$\int_\alpha^\beta g_1 df + \int_\alpha^\beta g_2 df = \int_\alpha^\beta [g_1 + g_2] df. \tag{5}$$

In particular, if $g_1 = g$ on $\alpha < x \leqq \gamma$, $g_1 = 0$ elsewhere, and $g_2 = g$ on $\gamma < x \leqq \beta$, $g_2 = 0$ elsewhere, γ being any point on $\alpha < x \leqq \beta$, it then follows from equations (4) and (5) that

$$\int_\alpha^\gamma g df + \int_\gamma^\beta g df = \int_\alpha^\beta g df, \tag{6}$$

where each interval over which the integral is taken is open on the left. If γ is a point of discontinuity of f and $g(\gamma) \neq 0$, then $\int_\gamma g df \neq 0$ and consequently cannot be included twice if (6) is to hold.

When f is a monotone function such that $f(\alpha+0) = 0$ and $f(\beta) = 1$, the integral gives the average value of g and hence is equal to a number μ lying between the maximum M and the minimum m of g. When the conditions $f(\alpha+0) = 0$ and $f(\beta) = 1$ are not satisfied, it follows from (3) that the number μ must be multiplied by $f(\beta) - f(\alpha+0)$ to give the value of the integral. Thus

$$\int_\alpha^\beta g(x) df(x) = \mu[f(\beta) - f(\alpha + 0)]. \tag{7}$$

It is easily seen that (7) is true for an arbitrary function of limited variation. By means of equations (6) and (7) it can readily be proved that the integral which we have defined reduces to the classical Stieltjes integral when g is continuous.

We shall show that the formula for integration by parts,

$$\int_\alpha^\beta d[f(x)g(x)] = \int_\alpha^\beta g(x) df(x) + \int_\alpha^\beta f(x) dg(x), \tag{8}$$

is valid provided the functions f and g are of limited variation and their determinations at points of singularity are such that

$$f(c) = [f(c + 0) + f(c - 0)]/2,$$
$$g(c) = [g(c + 0) + g(c - 0)]/2,$$

for every c of (α, β). First let us note that since f and g are of limited variation, their singularities are of the first kind only. Hence we are assured of the existence of the integrals of equa-

tion (8). Next if (a, b)* is any sub-interval of (α, β), we have the equations

$$\int_a^b d[f(x)g(x)] - \int_a^b g(x)df(x) - \int_a^b f(x)dg(x) = f(b)g(b) - f(a)g(a) - \nu[f(b) - f(a)] - \mu[g(b) - g(a)], \tag{9}$$

where $f(a) \leqq \mu \leqq f(b)$ and $g(a) \leqq \nu \leqq g(b)$,

$$0 = f(b)g(b) - f(a)g(a) - \frac{f(a) + f(b)}{2}[g(b) - g(a)] - \frac{g(a) + g(b)}{2}[f(b) - f(a)]. \tag{10}$$

Subtracting equation (10) from equation (9) we obtain

$$\int_a^b d[f(x)g(x)] - \int_a^b g(x)df(x) - \int_a^b f(x)dg(x) = \phi[g(b) - g(a)] + \gamma[f(b) - f(a)], \tag{11}$$

where $\phi = [f(b)+f(a)]/2-\mu$ and $\gamma = [g(b)+g(a)]/2-\nu$. When the integration is extended over a single point c (that is, from $c-0$ to $c+0$), then the numbers μ and ν of equation (9) become respectively $f(c)$ and $g(c)$. Hence $\phi = \gamma = 0$ by virtue of the determinations of $f(x)$ and $g(x)$ at their points of discontinuity. It follows that the formula for integration by parts is valid provided the interval of integration is a single point.

Returning to the general interval (a, b), we observe that $|\phi|$ and $|\gamma|$ are less than or equal to the oscillations of $f(x)$ and $g(x)$ in (a, b). Hence if c is any point of (α, β), we can find an interval (a, b) including c and such that $|\phi|$ and $|\gamma|$ are less than $\epsilon/2$ for each of the intervals $(a+0, c-0)$ and $(c+0, b+0)$. Thus

$$\left| \int_a^b d[f(x)g(x)] - \int_a^b g(x)df(x) - \int_a^b f(x)dg(x) \right| < \epsilon(|f(b) - f(a)| + |g(b) - g(a)|). \tag{12}$$

We can form a finite decomposition of (α, β) such that for each interval (a, b) of this decomposition inequality (12) holds. It follows that the expression

* For the sake of simplicity a is taken as a point of continuity of f. When this is not the case the necessary modifications are obvious.

$$\left| \int_a^\beta d[f(x)g(x)] - \int_a^\beta g(x)df(x) - \int_a^\beta f(x)dg(x) \right|$$

is less than ϵ multiplied by the total variation of $f(x)$ plus the total variation of $g(x)$. Hence this expression must be equal to 0 and therefore the formula for integration by parts is valid under the above hypotheses.

UNIVERSITY OF MICHIGAN

A SET OF POSTULATES FOR BOOLEAN ALGEBRA

BY SOLOMON HOBERMAN AND J. C. C. MCKINSEY

1. *A New Set of Postulates.* In the development of a Boolean Algebra, Boole's Law of Development

$$f(x) = f(1)x + f(0)x',$$

stands out as a basic relationship. This law is so all embracing that the question naturally arises, if this is set as a postulate, what postulates in addition to it are needed to define a Boolean Algebra? Using as undefined a class K and the Sheffer stroke function, we shall show that, in addition to a form of Boole's Law, only two "trivial" postulates are required.

POSTULATES.*

I. *K contains at least two elements.*

II. *If a and b are elements of K, then a/b is an element of K.*

Definitions: $a' = a/a$, $a \cdot b = a'/b'$, and $a+b = (a/b)'$.

III. *There exists in K a unique element 0, such that, if $f(x)$ is any function definable in terms of/and elements of K, we have, for any x in K,*

$$f(x) = f(0')x + f(0)x'.$$

THEOREM 1. $0'' = 0$.

Proof: From III, and the preceding definitions, we have

$$x = 0'x + 0x' = [(0'x)/(0x')]'; \tag{1}$$

in particular

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* This is the smallest set of postulates for a Boolean Algebra yet given.

$$0 = [(0'0)/(00')]'.$$

Thus there exists in K an element 1, such that

$$0 = 1'.$$

From (1) above

$$1 = 0'1 + 01,$$

and

$$1 = 1''1 + 1'1'.$$

Letting $f(x) = x''x + x'x'$ in III, we have

$$x''x + x'x' = (0'''0' + 0''0'')x + (0''0 + 0'0')x';$$

in particular

$$1 = 1''1 + 1'1' = (0'''0' + 0''0'')1 + (0''0 + 0'0')1'.$$

This becomes

$$1 = 0'''1 + 0'1',$$

since letting $f(x)$ equal x'' and x' respectively in III yields

$$x'' = 0'''x + 0''x', \quad \text{and} \quad x' = 0''x + 0'x';$$

in particular

$$0''' = 0'''0' + 0''0'', \quad \text{and} \quad 0' = 0''0 + 0'0'.$$

Letting $f(x) = 0'''x + x''x'$ in III, we have

$$0'''x + x''x' = (0'''0' + 0'''0'')x + (0'''0 + 0''0')x';$$

in particular

$$1 = 0'''1 + 1''1' = (0'''0' + 0'''0'')1 + (0'''0 + 0''0')1'.$$

This becomes

$$1 = 0'''1 + 0''1',$$

since

$$0''' = 0'''0' + 0'''0'', \quad \text{and} \quad 0'' = 0'''0 + 0''0'.$$

From III, $x'' = 0'''x + 0''x'$; thus $1'' = 0'''1 + 0''1'$, and $1 = 1''$. Hence $1' = 1'''$ and $0 = 0''$.

THEOREM 2. $x'' = x$.

Proof: $x'' = 0'''x + 0''x' = 0'x + 0x' = x$.

From Theorem 1 and the definition of 1, we have $0' = 1$, and thus

$$0/0 = 0' = 1, \quad \text{and} \quad 1/1 = 1' = 0.$$

THEOREM 3. $1/0 = 0/1 = 1$.

Proof: From III and the definitions, we have

$$1 = 0'1 + 01' = 0/0 + 1/1 = 1 + 0 = (1/0)';$$
$$0 = 1/0.$$
$$0' = 0''0 + 0'0' = 1/1 + 0/0 = 0 + 1 = (0/1)';$$
$$0 = 0/1.$$

THEOREM 4. $0+0=0$, $1+0=1$, $0+1=1$, $1+1=1$, $00=0$, $10=0$, $01=0$, and $11=1$.

Proof: These equations follow immediately upon using the results of the preceding theorems in the definitions of $+$ and $\cdot$.

In the following theorems the equations are obtained by letting $f(x)$ equal the left-hand side.

THEOREM 5. $1x = x = x\,1 = 0 + x = x + 0$.

Proof:
$$1x = (1\ 1)x + (1\ 0)x' = 1x + 0x',$$
$$x = 1x + 0x',$$
$$x1 = (1\ 1)x + (0\ 1)x' = 1x + 0x',$$
$$0 + x = (0+1)x + (0+0)x' = 1x + 0x',$$
$$x + 0 = (1+0)x + (0+0)x' = 1x + 0x';$$

since all five are equal to $1x + 0x'$, the theorem follows.

THEOREM 6. $0x = x0 = 0 = xx'$.

Proof:
$$0x = (0\ 1)x + (0\ 0)x' = 0x + 0x',$$
$$x0 = (1\ 0)x + (0\ 0)x' = 0x + 0x',$$
$$0 = 0x + 0x',$$
$$xx' = (1\ 1')x + (0\ 0')x' = 0x + 0x';$$

since all four are equal to $0x + 0x'$, the theorem follows.

THEOREM 7. $1 + x = x + 1 = 1 = x + x'$.

Proof:
$$1 + x = (1+1)x + (1+0)x' = 1x + 1x',$$
$$x + 1 = (1+1)x + (0+1)x' = 1x + 1x',$$
$$1 = 1x + 1x',$$
$$x + x' = (1+1')x + (0+0')x' = 1x + 1x';$$

since all four are equal to $1x+1x'$, the theorem follows.

THEOREM 8. $ax=xa$.

Proof: $ax=(a\ 1)x+(a\ 0)x'=(1\ a)x+(0\ a)x'=xa.$

THEOREM 9. $a+x=x+a$.

Proof: $a+x=(a+1)x+(a+0)x'=(1+a)x+(0+a)x'$
$=x+a.$

THEOREM 10. $x+bc=(x+b)(x+c)$.

Proof: $(x+b)(x+c)=(1+b)(1+c)x+(0+b)(0+c)x'$
$=(1)(1)x+(b)(c)x'=1x+bc\ x'$
$=(1+bc)x+(0+bc)x'=x+bc.$

THEOREM 11. $xb+xc=x(b+c)$.

Proof: $xb+xc=(1b+1c)x+(0b+0c)x'=(b+c)x+0x'$
$=(b+c)x=x(b+c).$

The postulates we have given are known to be true in a Boolean Algebra, therefore they are necessary. We shall show that they are sufficient by showing that Huntington's postulates are derivable from them.

2. *Huntington's Postulates and their Derivation.* The following is Huntington's set of postulates; to each is appended a brief indication of its derivation from those of our set.

1. (a) *If a and b are elements of K, then $a+b$ is an element of K.* By definition, $a+b=(a/b)'=(a/b)/(a/b)$; by Postulate I, if a and b are elements of K, a/b is an element, and $a+b$ is an element.

(b) *If a and b are elements of K, then $a\cdot b$ is an element of K.* By definition, $a\cdot b=a'/b'=(a/a)/(b/b)$ is an element of K as above.

2. (a) *There exists an element* 0 *in K such that $a+0=a$.*

(b) *There exists an element* 1 *in K such that $a\cdot 1=a$.*
Theorem 5.

3. (a) *If a, b, $a+b$, $b+a$ belong to K, then $a+b=b+a$.*
Theorem 9.

(b) *If a, b, ab, ba belong to K, then $ab=ba$.*
Theorem 8.

4. (a) *If a, b, c, bc, $a+bc$, $a+b$, $a+c$, and $(a+b)(a+c)$ belong to K, then $a+bc=(a+b)(a+c)$.*
Theorem 10.

(b) *If* $a, b, c, b+c, a(b+c), ab, ac,$ *and* $ab+ac$ *belong to* K, *then* $ab+ac=a(b+c)$.
Theorem 11.

5. *If* 0 *and* 1 *exist and are unique, then for every element* a *belonging to* K *there exists an element* a' *in* K *such that* $a+a'=1$ *and* $aa'=0$.
Theorems 1, 6, and 7.

6. *There are at least two distinct elements in* K.
Postulate I.

3. *A Two Element Boolean Algebra.* A set of postulates for a two element Boolean Algebra can be obtained by changing Postulate I to: "K contains two and only two elements."

4. *Independence Examples*:

1. $K = \{\alpha\}$, $\quad \alpha/\alpha = \alpha.$

2. $K = \{0, \beta\}$,

$'$	0	β	γ	δ
0	δ	γ	β	0
β	γ	γ	0	0
γ	β	0	β	0
δ	0	0	0	0

3. $K = \{\alpha, \beta\}$,

$'$	α	β
α	α	α
β	α	α

NEW YORK UNIVERSITY

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STATE COLLEGE, PENNSYLVANIA, SUMMER MEETING AND COLLOQUIUM, September 7–10, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than August 3. Professor John von Neumann will deliver a series of colloquium lectures on *Continuous geometry*. By invitation of the Program Committee, Professor Hassler Whitney will give an address entitled *Topological properties of differentiable manifolds*.

NEW YORK CITY, October 30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than October 2. Abstracts received by the Secretaries not later than September 3 will appear in the September issue of the Bulletin. By invitation of the Program Committee, Professor D. J. Struik will deliver an address entitled *The application of tensor analysis to problems of electrical engineering*.

IOWA CITY, IOWA, November 26–27, 1937.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than October 30; they will appear in the November issue of the Bulletin. By invitation of the Program Committee, addresses will be delivered by Professor W. J. Trjitzinsky on *Singular point problems in the theory of linear differential equations*, and by Professor A. H. Copeland on *Fundamental concepts of the theory of probability*.

PASADENA, CALIFORNIA, November 27, 1937.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than October 30; they will appear in the November issue of the Bulletin.

INDIANAPOLIS, INDIANA, ANNUAL MEETING, December 28–30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than November 23, 1937. Abstracts received by the Secretaries not later than November 2 will appear in the November issue of the Bulletin.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to the Managing Editor, D. R. CURTISS, Northwestern University, Evanston, Ill. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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WHOLE NUMBER 455

CONTENTS

PAGE

Modern Methods of Analysis in Potential Theory. By G. C. Evans 481

The Existence of Algebraic Plane Curves. By T. R. Hollcroft 503

Jensen's Inequality. By E. J. McShane 521

Bilinear Functionals on the Space of Bounded, Measurable Functions. By H. H. Goldstine 528

On Locally Compact Metrisable Spaces. By H. E. Vaughan 532

On Some Gap Theorems for Euler's Method of Summation of Series. By Yoshitomo Okada 536

Note on the Continuity of the Ergodic Function. By M. H. Martin 541

Real Canonical Binary Symmetric Trilinear Forms. By Rufus Oldenburger 546

Some Inequalities Concerning Functions of Exponential Type. By R. J. Duffin and A. C. Schaeffer 554

Note on the Relation between Continuity and Degree of Polynomial Approximation in the Complex Domain. By J. L. Walsh and W. E. Sewell 557

On Bounded Convolutions. By E. R. Van Kampen and Aurel Wintner 564

On a Theorem of Higher Reciprocity. By Albert Whiteman 567

The Most General Transformations of Plane Regions which Transform Circles into Circles. By Constantin Carathéodory 573

A Note on Determinants. By M. H. Ingraham 579

A New Definition of a Stieltjes Integral. By A. H. Copeland 581

A Set of Postulates for Boolean Algebra. By Solomon Hoberman and J. C. C. McKinsey 588

For official Communications and Notes, see the inside of the back cover.

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